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OF

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BEGINNERS IN ENGINEERING

BY

MANSFIELD MERRIMAN

MEMBER OF AMERICAN SOCIETY OF CIVIL ENGINEERS

SEVENTH THOUSAND

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PREFACE

During the past forty years great advances have been made in the methods of instruction in all branches of applied mechanics, but little or no change has taken place in the manner of presenting the subject of rational mechanics. This elementary volume is an attempt to apply the best methods of applied mechanics to the development of the fundamental principles and methods of rational mechanics. To this end, constant appeals are made to experience, by which alone the laws of mechanics can be established, numerous numerical illustrations are given, many queries and problems are stated as exercises for the student, and a system of units is employed with which every boy is acquainted.

The field of rational mechanics is so vast that no book can present more than a part of the same. Even the limited course usually given in engineering colleges is so difficult and appeals so little to the students' experience that few acquire a complete mastery of it. In the opinion of the author there should be given in every engineering college two courses in rational mechanics, an elementary one in which only as much mathematics is employed as is indispensably necessary, and an advanced one after the completion of the course in calculus. It cannot be doubted that the fundamental principles and methods of a science are of greatest value and importance, and if no course in mechanics is given until calculus

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has been completed, as is now generally the case, the student is introduced to a wilderness of algebraic matter in which these principles are largely obscured. Fortunately the fundamental elements can be presented without such advanced mathematics, and this book is an attempt in that direction.

To read this volume with interest and profit, only a knowledge of plane geometry, elementary algebra, and plane trigonometry is required. It is intended for manual-training schools, for a first course in engineering colleges, and for young men in general who have the preparation just indicated. To all who may use the book, it is strongly recommended that many numerical problems should be solved, and that in so doing the actual forces and bodies should be always kept in mind with the principles that govern their relations. Forty lessons thoroughly mastered will form a solid substructure on which applied mechanics may safely stand. If this be accomplished and an advanced course be later pursued, as above advocated, it is believed that the interests of sound engineering education will be materially promoted.

M. M.

CONTENTS

CHAPTER I. CONCURRENT FORCES

PAGE
ARTICLE 1. DEFINITIONS. 7
2. AXIOMS OR LAWS. 11
3. COMPONENTS OF FORCES. 14
4. THE RESULTANT OF FORCES. 17
5. CONDITIONS OF EQUILIBRIUM. 21
6. THE PARALLELOGRAM OF FORCES. 24

CHAPTER II. PARALLEL FORCES

ARTICLE 7. DEFINITIONS AND AXIOMS. 29
8. THE PRINCIPLE OF MOMENTS. 33
9. THE RESULTANT. 36
10. COUPLES. 40
11. NON-CONCURRENT FORCES. 43
12. PARALLEL FORCES IN SPACE. 48

CHAPTER III. CENTER OF GRAVITY

ARTICLE 13. DEFINITIONS AND PRINCIPLES. 51
14. CENTERS OF GRAVITY FOR LINES. 54
15. CENTERS OF GRAVITY FOR SURFACES. 57
16. STABLE AND UNSTABLE EQUILIBRIUM. 63
17. STABILITY AGAINST ROTATION. 67

CHAPTER IV. RESISTANCE AND WORK

ARTICLE 18. RESISTING FORCES. 72
19. FRICTION. 75
20. STABILITY AGAINST SLIDING. 80
21. GRAVITY AND WORK. 84
22. WORK AGAINST FRICTION. 88

CHAPTER V. SIMPLE MACHINES

PAGE
ARTICLE 23. THE SIMPLE LEVER. 94
24. SYSTEMS OF LEVERS. 98
25. THE INCLINED PLANE. 101
26. THE SCREW. 105
27. THE PULLEY. 108
28. EFFECT AND EFFICIENCY. 111

CHAPTER VI. GRAVITY AND MOTION

ARTICLE 29. VELOCITY AND ACCELERATION. 115
30. VERTICAL FALL OF BODIES. 119
31. OBLIQUE FALL. 123
32. POTENTIAL AND KINETIC ENERGY. 126
33. MOTION OF PROJECTILES. 131
34. COMPOSITION OF VELOCITIES. 136

CHAPTER VII. INERTIA AND ROTATION

ARTICLE 35. FORCE AND INERTIA. 141
36. WORK AGAINST INERTIA. 145
37. CENTRIFUGAL FORCE. 149
38. REVOLVING BODIES. 154
39. ROLLING BODIES. 158
40. PENDULUMS. 162

APPENDIX

ANSWERS TO PROBLEMS. 167
TRIGONOMETRIC FUNCTIONS. 170
INDEX. 171

ELEMENTS OF MECHANICS

CHAPTER I

CONCURRENT FORCES

ARTICLE 1. DEFINITIONS

Mechanics is the science that treats of forces and of their effects. Archimedes, a celebrated mathematician and engineer, announced some of its fundamental principles about 250 B.C., and these were greatly extended by Galileo and Newton in the seventeenth century. Mechanics is the foundation of modern engineering, and works on applied mechanics treat mainly of engineering problems. The principles and methods set forth in this elementary volume are the foundation of all branches of applied mechanics.

Force is manifested to our sense of feeling by pressure and by tension; when the hand pushes a body pressure is felt, when it pulls a body tension is felt. These applied forces of pressure and tension will cause the body to move if they are sufficient to overcome the resisting forces. Force may arise from muscular effort, from gravitation, from magnetic or electric action, from molecular attraction or repulsion, but elementary mechanics treats only of the pressure and tension which it causes and of the motions of bodies which it produces. This volume deals with solid bodies only.

A force has both magnitude and direction. Its magnitude may be measured in pounds, the unit of magnitude being the force exerted by gravity in London upon the standard block of platinum called the pound weight; or it may be measured in kilograms, the unit being the force exerted by gravity in Paris upon the standard block called the kilogram weight. Its direction is indicated by the statement as to whether it acts toward or away from a given body, and by the angle which its line of action makes with a fixed line of reference.

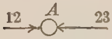

FIG. 1

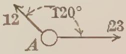

FIG. 2

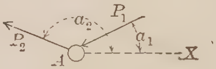

FIG. 3

A force acting on a body may be represented on a diagram by a straight line, the length of the line denoting its magnitude, and the position of that line with respect to the body showing its line of action, while an arrowhead gives its direction in that line of action. Thus, if two forces of 12 and 23 pounds act toward a body A in opposite directions, they may be represented as in Fig. 1; if they act away from the body in directions differing by 120 degrees, they may be represented as in Fig. 2. In Fig. 3 the pressing force P_1 acts toward the body and the pulling force P_2 acts away from it, the former making the angle a_1 and the latter the angle a_2 with the fixed line of reference AX.

The 'component' of a force in a given direction is defined as a force represented on the given line of direction by the intercept included between normals drawn to it from the ends of the force-line. Thus, AB in Fig. 4 represents the component of the force P in the direction AX. If a be the angle between the direction of P and that of AX, the magnitude of the component is $P \cos a$. When the angle a is zero, then AB is equal to P; when a is a right angle, as in Fig. 5, then AB is zero. Usually the given line of direction is taken as either horizontal or vertical, and the component is called the horizontal or vertical component of the force, as the case may be.

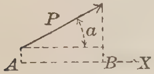

FIG. 4

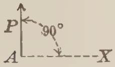

FIG. 5

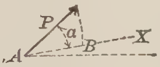

FIG. 6

Forces acting upon a body are said to be 'in equilibrium' when they are so arranged in magnitude and direction that the body remains at rest. The simplest case is that of a body on a horizontal table, the force of gravity urging it downward and the pressure of the table pushing it upward; since the body remains at rest these two forces are in equilibrium. In Figs. 1, 2, 3, the forces acting on the body A are evidently not in equilibrium.

'Concurrent forces' acting on a body are those having lines of direction which all meet at the same point. This case is of great importance in engineering, and its fundamental principles are set forth in this chapter, with respect to forces acting in the same plane.

The following problems should now be solved by the student, the diagrams and solutions being neatly recorded in ink in a special note-book devoted to this purpose. The three-place trigonometric table at the end of this volume is to be used when necessary and the final results of the computations should be expressed by three significant figures only.

PROBLEMS

1. What force must be applied in Fig. 1 to prevent motion?

2. Is it possible to produce equilibrium in Fig. 2 by applying another force? If so, give a sketch showing its approximate magnitude and direction.

3. If the angles a_1 and a_2 in Fig. 3 are each equal to a right angle, in what direction will the body move when P_1 is 6 and P_2 is $6\frac{1}{2}$ kilograms?

4. In Fig. 4 what is the component AB when the force is 100 pounds and the angle a is 29 degrees? also when the angle a is 134 degrees?

5. In Fig. 2 what force acts on the body in a direction toward the top of the page?

6. If P in Fig. 6 is 100 pounds, what is its component in the direction OX when the angle a is 30 degrees? also when a is 60 degrees? also when a is 120 degrees?

7. Draw Fig. 3 so that $a_1 = 90$ and $a_2 = 180$ degrees, while $P_1 = 43$ and $P_2 = 32$ pounds. What is the magnitude and direction of the force acting toward the foot of the page?

8. In Fig. 5 suppose that P is 14.1 pounds acting northward, and that the direction AX is eastward, the plane of the paper being horizontal. Compute the component of P which acts in a northwest direction.

9. Draw six equal concurring forces, each making an angle of 60 degrees with the next one. Are these forces in equilibrium? Why?

10. When only two forces act upon a body, show that motion will occur unless their magnitudes are equal, their directions opposite, and their lines of action in the same straight line.

ART. 2. AXIOMS OR LAWS

The science of mechanics, like that of geometry, is founded upon axioms or laws, the truth of which is known from the universal experience of mankind. The following axioms are needed in this chapter, and others will be found in Arts. 7 and 34.

> Axiom 1. When only one force acts upon a body it moves in a straight line in the direction of that force.
>
> Axiom 2. When two forces act upon a body it cannot remain at rest unless they are equal in magnitude and exactly opposite in direction.
>
> Axiom 3. When a body is at rest under several forces, each force is counteracted in its line of action by an equal and opposite force.
>
> Axiom 4. The effects produced by different forces in their lines of action are proportional to the magnitudes of the forces.
>
> Axiom 5. A single force exerts the greatest effect in its line of action, and it exerts no effect at right angles to its line of action.
>
> Axiom 6. The effect of a single force in a direction different from that of its line of action is the same as that of the component of the force in the given direction.

At the first reading some of these axioms may not be entirely clear to the student; if so, it is because his experience is limited and he should satisfy himself of their truth by trying simple experiments. Two threads attached to a small ball constitute an apparatus which will fully illustrate the truth of the first, second, and fifth axioms. The word 'effect' here means the pressure or tension that is caused by the forces.

The third axiom is the converse of the second and expresses the law stated by Newton under the form "to every action there is always opposed an equal reaction." Thus, if a man pulls upon a rope attached to a post, the post pulls upon the rope by the same amount and in the opposite direction. Again, if in Fig. 7 the three forces are in equilibrium, F_1 and F_2 produce a force which is equal and directly opposite to F; also F_2 is equal and directly opposite to a force caused by F and F_1.

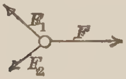

FIG. 7

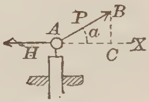

FIG. 8

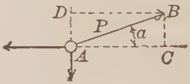

FIG. 9

The fourth axiom may perhaps not be immediately accepted, because in physiology effects are often not proportional to the magnitude of their causes. In the mechanics of rigid bodies, however, no exception has ever been found to this law. Thus, if the body in Fig. 7 remains at rest, one pound added to F always produces the same increase in tension in F_1 and F_2 whatever be the magnitude of these forces before the addition was made. Or, if several forces act upon a body in the same direction, each producing a pressure, the sum of these effects is the final pressure.

The sixth axiom or law is less evident than any of the others, but all experiments and all experience prove its truth. Fig. 8 shows a ring free to move in a vertical plane around the point where it is fastened to the vertical post. Let this ring be pulled by the horizontal force H and by the force P which is inclined at the angle a to the horizontal. If these forces are measured by spring balances, it is found that their ratio is the same as that of AC to AB; or if P is represented by AB, then will the force which is equal and opposite to H be represented by AC, and hence the horizontal pull exerted by P is equal to its horizontal component. Similarly in Fig. 9 if AB represents the magnitude of P, then it is found that the horizontal pull is represented by AC and the vertical pull by AD, and these represent the horizontal and vertical components of P.

When a body is free to move, the word 'effect' means the velocity which the body acquires under the action of the forces, and the above axioms apply also, as will be illustrated in Art. 34. The ball and string will also be useful in making experiments where motion occurs.

PROBLEMS

11. If in Fig. 7 the force F is 50 pounds while F_1 and F_2 are each 25 pounds, what angles must F_1 and F_2 make with F in order to secure equilibrium?

12. What happens to the body in Fig. 7 if F is 10 pounds while F_1 and F_2 are each 3 pounds?

13. If F_2 in Fig. 7 is zero, what relations are necessary between the magnitudes and directions of F_1 and F in order to secure equilibrium?

14. What horizontal pull does P produce in Fig. 8 if its magnitude is 100 pounds and its angle of inclination is 31 degrees? What vertical pull does it produce?

15. If H is 50 pounds in Fig. 8, what is the value P when the angle a is 44° 45′? also when the angle a is 1° 15′?

16. What horizontal and vertical forces must be applied to the body in Fig. 9 in order to prevent motion, if P be 1000 pounds and a be 45° 15′?

17. Seven men pull on a rope, each exerting the same force, and the total tension is 490 pounds. What is the tension when three of the men stop pulling?

18. Compute the horizontal and vertical components of a force of 1000 pounds when it makes an angle of 95° 30′ with a horizontal line.

19. In a horizontal plane a force of 33 pounds acts eastward and is held in equilibrium by two equal forces acting northwestward and southwestward. Find each of these forces.

20. What is the tension in a rope when one end is tied to a post and a man pulls on the other end with a force of 65 pounds? What is the tension in a rope when two men each exerting 65 pounds, pull at opposite ends?

ART. 3. COMPONENTS OF FORCES

Considering concurrent forces in one plane only, let them be represented by lines acting away from the body, and let the paper be held in a vertical plane. Forces acting upward or downward may then be called 'vertical forces' and those at right angles to them may be called 'horizontal forces.' Vertical forces and components are taken as positive when acting upward and as negative when acting downward; horizontal forces and components are taken as positive when acting toward the right and as negative when acting toward the left. Thus, in Fig. 10, if the vertical forces V_1 and V_2 are 27 and 19 kilograms, they may be written $+27$ and -19, the sign $+$ meaning that the former acts upward, while the sign $-$ means that the latter acts downward.

When a force P acts upon a body A, in a direction making an angle a with a horizontal line, its vertical and horizontal components are represented by the distances AC and AD in Fig. 9. The horizontal component may be called H and the vertical component may be called V. Since ABC is a right-angled triangle, these components may be expressed by the formulas

$$H = P \cos a \qquad\qquad V = P \sin a$$

from which numerical values may be computed.

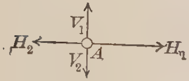

FIG. 10

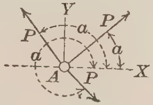

FIG. 11

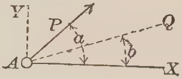

FIG. 12

One of the most common problems in mechanics is that of the discussion of concurrent forces by the help of their components. In Fig. 11 let AX be a horizontal line and AY a vertical line drawn through a body A subject to concurrent forces, each of which acts away from the body. Let P be any force, and a the angle measured from AX around to the line of action of P. Then $P \cos a$ is the horizontal component of P, which is positive or negative according as $\cos a$ is positive or negative; also $P \sin a$ is the vertical component of P, which is positive or negative according as $\sin a$ is positive or negative. For instance, if P is 100 pounds and a is 120 degrees, then the horizontal component is -50 pounds and the vertical component is $+86.6$ pounds; if a is 315 degrees, then the horizontal component is $+70.7$ pounds and the vertical component is -70.7 pounds.

When the components H and V are given, the values of P and a may be easily derived from the right-angled triangle ABC in Fig. 9. Since AC^2+BC^2 equals AB^2, and since BC/AC is the tangent of the angle a, it follows that

$$P^2 = H^2 + V^2 \qquad \text{and} \qquad \tan a = V/H$$

For example, let H be $+5$ and V be $+12$ pounds, then P is 13 pounds and $\tan a$ is $12/5$ or $+2.40$, whence from a trigonometric table a is found to be about $67\frac{1}{3}$ degrees. Again, if H be -75 and V be $+37\frac{1}{2}$ pounds, then P is 83.85 pounds and $\tan a$ is -0.50, whence, by help of the table, a is found to be $153° 26'$.

Another problem is that of finding the component in a direction not horizontal or vertical. In Fig. 12 let P be a force making an angle a with the horizontal, and let it be required to find its component in the direction AQ which makes an angle b with the horizontal. By the definition (Art. 1) this component is $P\cos(a-b)$. For example, if P be 100 pounds, a be 64 and b be 31 degrees, the component in the direction AQ is $100\cos 33°$ or 83.9 pounds. The component perpendicular to the direction AQ is $P\sin(a-b)$, which for the same data is $100\sin 33°$ or 54.5 pounds.

Before beginning the numerical solution of any problem the student should always draw a diagram representing the given data. Unless otherwise stated the forces are to be understood as acting away from the body and the arrow-heads should be drawn accordingly. The arbitrary use of formulas is to be avoided, and each problem should be worked out from reasoning based on its diagram.

PROBLEMS

21. In Fig. 10 let the values of H_1 and H_2 be $+75$ and -50 pounds. What force moves the body horizontally?

22. In Fig. 11 draw a force of 125 pounds acting away

from the body and making an angle of 240° 15′ with AX. Compute its horizontal and vertical components.

23. If P be 100 kilograms and a be 270 degrees, what are the vertical and horizontal components?

24. Draw a force of 10 pounds whose horizontal component is $+5$ pounds, and whose vertical component is -8.66 pounds. What angle does it make with the horizontal?

25. A body weighing 2 pounds rests on a table and is acted upon by a force of 8 pounds making an angle of 30° 15′ with the horizontal. What is the total pressure on the table?

26. In Fig. 12 compute the component in the direction AQ when a is 93°, b is 3°, and P is 328 pounds.

27. Draw Fig. 12 so that a is 135° and b is 146°, and compute the component in the direction AQ when P is 7.5 pounds.

28. If the horizontal and vertical components of a force are -34.6 and $+93.8$ pounds, what is the magnitude of the force? What angle does it make with the horizontal?

29. Find the magnitude of a force making an angle of 99 degrees with the horizontal when it is known that its vertical component is 24.7 kilograms.

30. In Fig. 12 let $P = 100$ pounds and $b = 13$ degrees. What is the value of a when the component in the direction AQ is $60\frac{1}{2}$ pounds?

ART. 4. THE RESULTANT OF FORCES

The 'resultant' of a system of concurrent forces acting on a body is a single force that produces the same effect in its direction as the given forces. If this resultant is 0, the forces are in equilibrium and the body remains at rest; if a resultant force exists, the body moves in the direction of that resultant. (Axiom 1, Art. 2.)

When several forces act upon a body A in the same line and direction the sum of these (Axiom 4, Art. 2) is their resultant. If some act in the opposite direction to others, the algebraic sum is the resultant. Thus in Fig. 1 the resultant is $+12-23=-11$, the minus sign showing that the resultant acts toward the left.

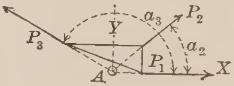

FIG. 13

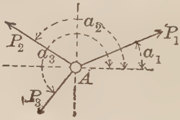

FIG. 14

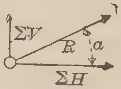

FIG. 15

Horizontal components of concurrent forces are treated exactly like forces acting in the same line, their algebraic sum giving the resultant horizontal component, while the algebraic sum of the vertical components gives the resultant vertical component. For example, let it be required to find the resultant horizontal component of the three concurrent forces acting upon the triangle in Fig. 13, their magnitudes being $P_1=65$, $P_2=31$, $P_3=100$ pounds, and P_1 being horizontal, while the angles are $a_2=37° \ 15'$ and $a_3=153° \ 45'$. By the method of Art. 3, the separate horizontal components are

$$H_1=P_1 \cos 0° = 65(+1.000)=+65.0 \text{ pounds},$$
$$H_2=P_2 \cos a_2 = 31(+0.796)=+24.7 \text{ pounds},$$
$$H_3=P_3 \cos a_3 = 100(-0.897)=-89.7 \text{ pounds},$$

and the algebraic sum of these is 0.0, showing that the body has no tendency to move in a horizontal direction.

Let three component forces P_1, P_2, P_3, as in Fig. 14, act upon a body, the angles which they make with the horizontal being a_1, a_2, a_3. Their horizontal and vertical components are

$$H_1=P_1 \cos a_1 \qquad H_2=P_2 \cos a_2 \qquad H_3=P_3 \cos a_3$$
$$V_1=P_1 \sin a_1 \qquad V_2=P_2 \sin a_2 \qquad V_3=P_3 \sin a_3$$

and the algebraic sum of the horizontal components is the resultant horizontal component, while the algebraic sum of the vertical components is the resultant vertical component. Let the first algebraic sum be denoted by ΣH and the second be denoted by ΣV, or

$$H_1+H_2+H_3=\Sigma H, \qquad V_1+V_2+V_3=\Sigma V$$

Let the values of ΣH and ΣV be computed and be laid off as in Fig. 15. Then, as in Art. 3, it is clear that the final resultant R and the angle a which it makes with the horizontal are given by

$$R^2=(\Sigma H)^2+(\Sigma V)^2 \qquad \tan a=\Sigma V/\Sigma H$$

and thus the resultant of the three forces P_1, P_2, P_3 is fully determined in magnitude and direction.

As a numerical example let the following be the data for three concurrent forces:

$$P_1=200, \qquad P_2=150, \qquad P_3=250 \text{ pounds}$$
$$a_1=59^\circ 45', \qquad a_2=119^\circ 45', \qquad a_3=249^\circ 30'$$

The computation of the values of ΣH and ΣV may be arranged as follows:

$$\begin{array}{lr@{\qquad}lr}
H_1=200(+0.504)= & +100.8 & V_1=200(+0.864)= & +172.8 \\
H_2=150(-0.496)= & -74.4 & V_2=150(+0.868)= & +130.2 \\
H_3=250(-0.350)= & -87.5 & V_3=250(-0.937)= & -234.2 \\
\hline
\Sigma H= & -61.1 & \Sigma V= & +68.8
\end{array}$$

The body is hence urged toward the left by a force of 61.1 pounds, and upward by a force of 68.8 pounds. The magnitude of the resultant and its direction are next found:

$(\Sigma H)^2 = 3733$
$(\Sigma V)^2 = 4733$
$R^2 = 8466$
$R = 92.0$ pounds

$\Sigma V/\Sigma H = 68.8/61.1$
$\tan a = -1.126$
$a = 131° \; 37'$

CHECK.
$\sin a = 0.748$
$R \sin a = +68.8$

Here the last column gives a check on the computations. After a has been found, $\sin a$ is taken from the table and multiplied by the value of R; the product should be the same as the value of ΣV. Another check may be applied by computing $R \cos a$, which must be equal in value to ΣH.

In using a three-place trigonometric table, like that given at the end of this volume, the final results of computations cannot be relied upon beyond the third significant figure, since the last figure in the tables may be uncertain by nearly half a unit. To make the third significant figure of the final results as reliable as possible it is best to carry the intermediate results to four significant figures, as has been done above, but more than four significant figures should not be retained, as they add nothing to the precision of the results.

PROBLEMS

31. If the three angles in Fig. 7 are equal, what is the value of F when $F_1 = F_2 = 100$ kilograms?

32. Compute the resultant for Fig. 10 if V_1 and H_1 are each 301 pounds and V_2 and H_2 are each 291 pounds.

33. Compute the resultant vertical component for Fig. 13 with the data given above.

34. For Fig. 14 compute the resultant horizontal component from the following data:

$P_1 = 200$ pounds
$a_1 = 144° \; 30'$

$P_2 = 150$ pounds
$a_2 = 234° \; 00'$

$P_3 = 250$ pounds
$a_3 = 3° \; 45'$

35. For the same data compute the resultant vertical component.

36. For the same data compute the final resultant, its angle of direction, and check the work.

37. If $\sin a = 0.300$, find the values of $\cos a$ and $\tan a$ without using the table.

38. What is the value of R when $\Sigma V = +500$ pounds and $a = 150°$?

39. The magnitudes of three concurrent forces are 25, 60, and 65 pounds. The first is horizontal, the second vertical, and the third inclined to the horizontal at an angle whose sine is $-12/13$ and whose tangent is $+12/5$. Are the forces in equilibrium?

40. In a horizontal plane four concurrent forces act on a body, 19 pounds acting northeast, 48 pounds north, 12 pounds west, and 60 pounds south. Compute the magnitude and direction of their resultant.

ART. 5. CONDITIONS OF EQUILIBRIUM

When any number of concurrent forces act on a body, each may be replaced by its horizontal and vertical components. The algebraic sums of these and the resultant R can then be found, as in Art. 4. If R has any value different from 0, the body moves in the direction of R. If $R = 0$, the forces are in equilibrium and the body remains at rest. The condition for the equilibrium of concurrent forces hence is that these forces shall have no resultant.

From Art. 4 it is seen that a resultant R exists whenever both ΣH and ΣV have values different from 0. Hence the statement that $R = 0$ is the same as

$$\Sigma H = 0 \qquad \text{and} \qquad \Sigma V = 0$$

which are algebraic expressions for the conditions of equilbrium of concurrent forces. The first of these states that the algebraic sum of the horizontal components of the forces must be zero, and the second states that the algebraic sum of the vertical components of the forces must be zero. When both of these conditions are fulfilled the forces are in equilibrium and the body remains at rest; when one or both are not fulfilled the forces are not in equilibrium and the body moves.

When several concurrent forces are in equilibrium each force is equal in magnitude to the resultant of the other forces, but it acts in the opposite direction, for this is merely another statement of the third axiom of Art. 2. Thus in Fig. 14, the resultant of P_1 and P_2 is equal to P_3, but it acts in the opposite direction; the resultant of P_2 and P_3 is equal to P_1, but it acts in the opposite direction.

When it is known that a body is at rest the above conditions of equilibrium may be used to find one of the forces when the others are given. For example, in Fig. 16 let $H_1=+6200$, $H_2=-7300$, $V_1=+4900$, and $V_2=-7500$ pounds, and let S be an unknown force acting away from the body and holding them in equilibrium; since $S \cos a$ and $S \sin a$ are its horizontal and vertical components, the above conditions become

$$+6200-7300+S\cos a=0 \qquad +4900-7500+S\sin a=0$$

from which $S\cos a=+1100$ and $S\sin a=+2600$. As these are both positive, it is seen that a is less than 90°, and $S\sin a/S\cos a=+2.364=\tan a$, whence from the table a is found to be 67°. Also $S^2=1100^2+2600^2$, whence $S=2820$ pounds.

In the computation of stresses in bridge trusses, the angle a is generally known and it is required to find S. Fig. 17 shows forces or stresses acting at a joint of the lower chord of a Howe truss, all being given in direction and three in magnitude. The first condition of equilibrium for these concurrent forces is

$$+125\,000 - S_1 \cos 55^\circ - 93\,000 = 0$$

from which $S_1 = 55\,750$ pounds. The second condition of equilibrium is

$$S_2 - 55\,750 \cos 35^\circ - 4000 = 0$$

from which $S_2 = 49\,660$ pounds. As a check on the work the components of all the forces on the direction of S_1 may be found and added together, thus:

$$(125\,000 - 93\,000) \cos 55^\circ + (49\,660 - 4000) \cos 35^\circ - 55\,750$$

The sum of these should be 0, and on computation it is found to differ from this by only about 10 pounds, which is as close an agreement as can be expected with the use of three-place tables.

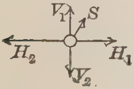

FIG. 16

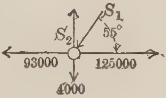

FIG. 17

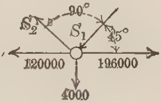

FIG. 18

PROBLEMS

41. Find the magnitude and direction of the resultant of the two forces in Fig. 2.

42. If a force is to be applied in Fig. 2 to hold the two forces in equilibrium, what is its magnitude and direction?

43. A horizontal and a vertical force are to be applied in

Fig. 4 to secure equilibrium. What are their magnitudes and directions?

44. If F_1 and F_2 are equal in Fig. 7 and a be the angle between them, find the magnitude of each for equilibrium.

45. In Fig. 9 let H be 42 kilograms and a be 30° 30′. Find the values of V and P.

46. If the magnitudes of the forces given in Art. 4 for Fig. 14 be doubled, what is the magnitude and direction of the resultant?

47. Nine concurring forces in a horizontal plane have a resultant which acts northwest by north. What is the direction of the force that will hold the nine given forces in equilibrium?

48. For Fig. 18 write out the two conditions of equilibrium.

49. Fig. 18 represents forces acting at a joint of the lower chord of a Warren truss. Find the unknown forces S_1 and S_2.

50. Write for Fig. 18 an expression for the sum of the components of all the forces on a line making an angle of 45° with the horizontal, and compute S_1 from it.

ART. 6. THE PARALLELOGRAM OF FORCES

Since forces can be represented in magnitude and direction by straight lines, it is possible to solve many problems on the drawing-board, thus avoiding numerical computations. The force-lines are to be drawn in the given directions and their magnitudes laid off to scale, and then by certain operations their resultant may be found or other required results be deduced.

The 'parallelogram of forces' is the name given to the fundamental principles of these graphic operations. It is as follows: If two forces P and Q acting on a body be represented by two lines meeting at a point, then their resultant is represented by the diagonal of the parallelogram formed on the two force-lines. Thus, in Figs. 19 and 20, let *op* and *oq* represent the magnitudes and directions of two forces *P* and *Q* acting on a body at *o*, and let the parallels *pr* and *qr* be drawn completing the parallelogram *oprq*; then the diagonal *or* represents the magnitude and direction of the resultant of the forces *P* and *Q*. To prove this let *op* be taken as horizontal, let *a* be the given angle *poq*, and *b* be the unknown angle *por*.

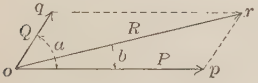

FIG. 19

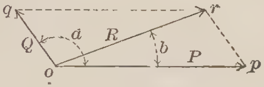

FIG. 20

Then the resultant horizontal and vertical components of the given forces are

$$H=P+Q\cos a \qquad V=Q\sin a$$

and hence, from Art. 4, the magnitude and direction of the resultant are determined by the expressions

$$R^2=P^2+Q^2+2PQ\cos a \qquad \tan b=Q\sin a/(P+Q\cos a)$$

But, by geometry, since the lines *oq* and *pr* are equal and parallel,

$$\overline{or}^2=\overline{op}^2+\overline{oq}^2+2op\,.\,oq\cos a \qquad \tan b=oq\sin a/(op+oq\,.\,\cos a)$$

which are the same as before. Therefore *or* represents the resultant of the two forces *P* and *Q*.

The same principle may be applied to find the resultant of three concurrent forces P_1, P_2, P_3, as shown in Figs. 21 and 22. First, consider only P_1 and P_2 and let a line be drawn from the end of each parallel to the other, then the line drawn from the point of intersection to the body gives R_1 as their resultant. Secondly, the force P_3 and the resultant R_1 are combined in the same manner to obtain R_2, which is the final resultant of P_1, P_2, P_3. If four or more forces are given, the same process may be continued.

Since a force equal and opposite to the resultant holds the others in equilibrium, the same method may be used to find one of several concurrent forces in equilibrium when the others are given.

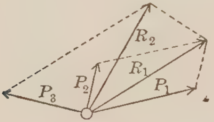

FIG. 21

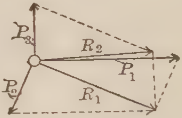

FIG. 22

If a single force be given and it be required to find two forces acting in given directions which will hold it in equilibrium, this can be done by the principle of the parallelogram of forces. Let *or* in Figs. 23, 24, and 25 represent the given force in magnitude and position, and let *om* and *on* be the lines along which the unknown forces act. From *r* draw lines parallel to these directions, thus determining the points *p* and *q*. Then *op* and *oq* represent the magnitudes of the forces which will hold *or* in equilibrium. If *or* acts away from the body, *op* and *oq* act toward it; if *or* acts toward the body, *op* and *oq* act away from it in order that equilibrium may obtain.

The process of finding two forces to replace or hold in equilibrium a single given force is called 'resolution'; this may be done by computation, using the properties of the triangle or parallelogram, and also by the conditions of equilibrium in Art. 5. As an example of the first method, let *or* in Fig. 23 represent the magnitude of the given force *R*, and let it be required to find two forces *P* and *Q*, acting in directions that make the angles *b* and *c* with *or*, which will replace *R*. Constructing the parallelogram *oprq*, it is seen that *op* and *oq* represent *P* and *Q*, and from the triangles *orp* and *orq*,

$$P = R\frac{\sin c}{\sin(b+c)} \qquad Q = R\frac{\sin b}{\sin(b+c)}$$

from which *P* and *Q* may be computed. Or, if any three quantities in these equations be given, the other two quantities may be computed. These equations show that, if *c* is equal to *b*, the forces *P* and *Q* are also equal, each being $\frac{1}{2}R/\cos b$; if *b* and *c* be both nearly 90°, then both *P* and *Q* are very large compared with *R*. It thus appears that a small force *R* may require large pressures or tensions to hold it in equilibrium.

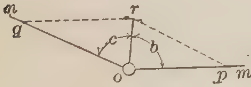

FIG. 23

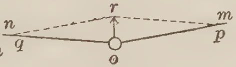

FIG. 24

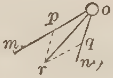

FIG. 25

From the above discussion, since *pr* equals *oq* in Fig. 23, it follows that three forces in equilibrium can always be represented by the three sides of a triangle. Thus if *AB* and *AC* be two sides of any triangle, the third side *BC* represents the magnitude of a force which will hold in equilibrium the two forces that are represented by *AB* and *AC*.

PROBLEMS

51. Draw Fig. 19 so that $P=6$ and $Q=8$ pounds, and $a=90$ degrees. Compute the values of R and b.

52. Draw Fig. 20 so that $P=6$ and $Q=8$ pounds, and $a=180$ degrees. What are the values of R and b?

53. If two forces, each of 1000 pounds, act away from a body, the angle between them being 178 degrees, compute the magnitude of force required to hold them in equilibrium.

54. For Fig. 23 show that $P/Q=\sin c/\sin b$.

55. From this formula find the value of c when $P=12$, $Q=13$ pounds, and $b=87°$.

56. Draw three concurrent forces $P_1=9$, $P_2=12$, $P_3=14$ pounds, acting toward a body and making the angles $a_1=30°$, $a_2=90°$, $a_3=240°$ with the horizontal. Construct their resultant and measure its magnitude and direction.

57. For Fig. 19 show that $R=P\cos b+Q\cos(a-b)$. Also show that $Q=R\sin b/\sin a$, and $P=R\sin(a-b)/\sin a$.

58. If two forces are equal, show that their resultant bisects the angle between them.

59. Two posts of equal height are 13 feet apart and a thin cord, 16 feet long, joins their tops. A load of 100 pounds hangs from the middle of this cord. Find the angle which each part of the cord makes with its post. Find the tension in each part of the cord.

60. In the last problem let the load hang at a distance of 5 feet from one of the posts. Compute the angle which each part of the cord makes with its post, and the tension in it.

CHAPTER II

PARALLEL FORCES

ART. 7. DEFINITIONS AND AXIOMS

Concurrent forces acting on a body cause it to move in the direction of their resultant, if a resultant exists. This motion is in a straight line (Axiom 1, Art. 2) since the resultant is equivalent to a single force acting upon the body. Such motion is often called one of translation.

Another kind of motion is that called rotation, where the body moves around an axis. In the case of a sphere or wheel, rotation may occur around a fixed axis, the body constantly remaining in the same portion of space, whereas in translation it occupies at each successive instant a new portion of space.

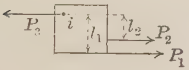

FIG. 26

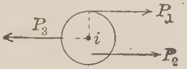

FIG. 27

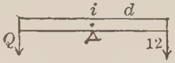

FIG. 28

An 'axis' is an imaginary fixed line about which a body rotates or tends to rotate under the action of forces. Thus, in Figs. 26-28 the axis is perpendicular to the plane of the paper and marked by the letter i. A force having its line of action passing through i evidently has no tendency to cause rotation around i. A force of given magnitude evidently has the greater tendency to produce rotation the further be its line of action from the axis; thus if P_1 and P_2 be equal in Fig. 27, the former has a greater effect than the latter.

The axioms stated in Art. 2 apply to parallel as well as to concurrent forces. There are, however, additional axioms or laws, applicable to both systems of forces, the truth of which has been established by universal experience and experiment:

> Axiom 7. A force whose direction passes through an axis has no tendency to cause a motion of rotation around that axis.
>
> Axiom 8. The tendency of a force in causing rotation is proportional to its magnitude and also proportional to the shortest distance from the axis to its line of action.

The student may convince himself of the truth of these axioms by attaching several strings to a piece of cardboard, as in Figs. 26 and 27, laying the same on a smooth table and using a pin for a vertical axis *i*. It will be found that no rotation of the card will result under the action of a single force P_3 when its line of action passes through *i*, and from Axiom 2 it is recognized that a force equal and directly opposite to P_3 is brought into action at the axis. If, however, the force P_1 or P_2 be applied so that its line of direction is at the distance l_1 or l_2 from the axis, rotation can easily be produced, and the more easily the greater be this distance.

Fig. 28 shows another simple method of illustrating Axiom 8. Let a bar be balanced upon a support at its middle and a spring balance be used to connect its left end to the floor so as to measure the force *Q*. If a load of 12 pounds be applied at the right end, the force *Q* will be found to be also 12 pounds; if, however, the same load be applied at *d*, half-way between *i* and the right end of the bar, the force indicated at *Q* will be 6 pounds.

The 'moment of a force with respect to an axis' is the product obtained by multiplying the magnitude of the force by the shortest distance from the axis to its line of action. This shortest distance is often called the 'lever arm' of the force. Thus if P be a force and l its lever arm, the product Pl is the moment of the force. By Axiom 8 the moment is a measure of the tendency of the force to cause rotation about the axis.

Unless otherwise stated the given forces are taken in the same plane and the axis is normal to that plane. If there be several forces P_1, P_2, P_3, with lever arms l_1, l_2, l_3, the moments of these forces with respect to the axis i are P_1l_1, P_2l_2, P_3l_3. Some of these may tend to cause rotation around i in one way and some in the opposite way. A moment tending to cause rotation like that of the hands of a clock is taken as positive, while one tending to cause rotation in the reverse way is taken as negative. Thus, in Fig. 27, the moment P_1l_1 is $+$, while P_2l_2 is $-$, and P_3 has no moment because it has no lever arm with respect to the axis i.

A moment is a compound quantity formed by multiplying a force by a distance. Its unit of measure is hence one pound-foot, if the force be measured in pounds and the lever arm in feet. Thus in Fig. 28, if Q be 6 pounds and its distance from i be 10 feet, its moment is -60 pound-feet; if a load of 12 pounds be at d, distant 5 feet from i, its moment is $+60$ pound-feet. Moments may also be expressed in pound-inches, kilogram-meters, or gram-centimeters, when the data are given in such units of measure.

PROBLEMS

61. In Fig. 27 let $P_1=12$, $P_2=18$ grams, and $l_1=6$, $l_2=4$ centimeters. What is the moment of each force with respect to i, and which way will the body turn?

62. For the above case what must be the value of P_3 in order that equilibrium may prevail?

63. In turning a grindstone what influence has the distance of the handle from the axis?

64. Let Fig. 28 represent a seesaw plank supported at its middle, and Q a man weighing 240 pounds. How many boys, weighing 80 pounds each, must be put at the right-hand end in order to balance the man?

65. Show that 10 pound-feet is equivalent to 138 254 gram-centimeters.

66. Draw a square with two forces acting in opposite directions along two opposite sides. What kind of motion will result? What forces must be added to prevent this motion?

67. A man carries a load at the end of a pole placed on his shoulder, balancing it by a downward pressure of his hand. If that downward pressure is one-third of the weight of the load, what are the relative distances of load and hand from the shoulder?

68. If the bar in Fig. 28 weighs 40 pounds and Q is 150 pounds, what is the vertical pressure upon the support when a load is put at d sufficient to cause equilibrium?

69. Draw Fig. 26 so that $l_1=8$ and $l_2=4$ centimeters, and let $P_1=27$ kilograms. Find P_2 so that equilibrium may occur, and draw it in the proper direction.

70. What happens if a support be placed under a plank at any other point than its middle? Why?

ART. 8. THE PRINCIPLE OF MOMENTS

From the definitions and axioms of Art. 7, it is now plain that, in Fig. 29, the tendency of the body to take a possible rotation is indicated by $P_1l_1 - P_2l_2$, since P_1l_1 tends to rotate it like the hands of a watch, while P_2l_2 has the opposite tendency. If P_1l_1 equals P_2l_2, there is no rotation about the axis i, and the body is in equilibrium. This important conclusion, often called the 'principle of moments,' may be formulated as follows:

> If forces acting in one plane upon a body are in equilibrium, the algebraic sum of their moments, about any point in the plane, is equal to zero.

The term "any point," in the enunciation of this law, is an important one. It refers not necessarily to an actual axis, but to any point arbitrarily chosen; for if the body be at rest, the forces have no tendency to cause motion about any point, and the principle is true for each and every point in the plane.

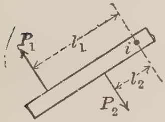

FIG. 29

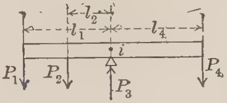

FIG. 30

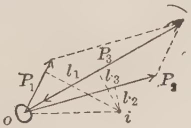

FIG. 31

This principle, although derived from the consideration of parallel forces, is equally true for concurrent forces. To show this let P_1, P_2, P_3, in Fig. 31, represent three concurrent forces in equilibrium, and let i be any point in the plane. Join oi and let a_1, a_2, a_3 be the angles which P_1, P_2, P_3 make with oi. Then, by Art. 5, the algebraic sum of the components of the forces in any direction is zero. Taking the line oi as horizontal in direction, the algebraic sum of the vertical components of the three forces is

$$P_1 \sin a_1 + P_2 \sin a_2 - P_3 \sin a_3 = 0.$$

Now let the normals l_1, l_2, l_3 be drawn from i to P_1, P_2, P_3; then $\sin a_1 = l_1/oi$, $\sin a_2 = l_2/oi$, $\sin a_3 = l_3/oi$. Substituting these in the equation, it reduces to

$$P_1 l_1 + P_2 l_2 - P_3 l_3 = 0$$

which agrees with the principle of moments, and this equation may be extended to include any number of concurrent forces in the same plane.

When a system of parallel forces is in equilibrium, their algebraic sum must be zero in order that no motion of translation may result. The algebraic sum of their moments about any point in the plane must also be zero in order that there may be no motion of rotation. Therefore, it follows that

$$\Sigma P = 0 \qquad\qquad \Sigma Pl = 0$$

are the conditions of equilibrium for parallel forces. Here ΣP means $P_1 + P_2 + P_3 +$ etc., and ΣPl means $P_1 l_1 + P_2 l_2 + P_3 l_3 +$ etc., the summation being algebraic in each case. Usually the forces are regarded as in a vertical plane, so that for ΣP upward forces are positive and downward ones are negative as in Art. 3, while for ΣPl the signs of the moments are positive or negative according to the convention in Art. 7.

These two conditions enable two unknown quantities to be determined when all the others are known. For instance, let two loads of 120 and 100 pounds be placed upon a beam, as shown in Fig. 32, and let it be required to find the upward pressures of the supports that will hold them in equilibrium. The first condition gives $P_1+P_2-120-100=0$. Taking an axis at the right end, the second condition gives $P_1\times 12-120\times 6-100\times 3=0$. From these two equations are found $P_1=85$ and $P_2=135$ pounds.

As a second example, take the bar in Fig. 33 and let it be required to find the force P_1 and the distance l_1 which are necessary for equilibrium. The first condition gives $+20+P_1-80-60=0$, from which $P_1=120$ pounds. The second condition, taking an axis at the left end, gives $80\times 4+60\times 12-120\times l_1=0$, from which $l_1=8\frac{2}{3}$ feet.

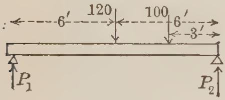

FIG. 32

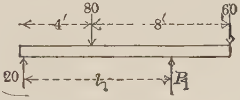

FIG. 33

In the solution of problems of equilibrium it is immaterial where the axis of moments be taken, as the principle of moments is true for every point in the plane. Thus, in the last example, if the axis be taken on the line of action of P_1, the equation is $20l_1-80(l_1-4)+60(12-l_1)=0$, from which $l_1=8\frac{2}{3}$ feet. It is usually preferable to take the axis on the line of action of one of the forces, as the moment of that force is then zero, and the equation is simplified.

PROBLEMS

71. Draw Fig. 26 so that $l_3=0$, $l_2=2$, and $l_1=2\frac{1}{2}$ inches. In which way will rotation occur when $P_3=6$, $P_2=5$, and $P_1=4$ pounds?

72. Which way will rotation occur in Fig. 30 when $l_1=8$,

$l_2=4$, $l_3=0$, $l_4=8$ inches, and $P_1=7$, $P_2=8$, $P_3=25$, $P_4=10$ pounds?

73. In Fig. 31 take a point on the line of direction of P_3 and write the equation of moments.

74. In Fig. 33 let $P_1=120$ pounds and $l_1=8$ feet. Will the body rotate in the positive or in the negative way?

75. In Fig. 9 show, by taking moments about C, that $H \sin a - V \cos a = 0$ is necessary for equilibrium.

76. Draw a beam with supports at its ends and a load of 60 pounds at 3 feet from the left end. If the length of the beam is 18 feet, find the upward pressures of the supports.

77. A wheelbarrow has a load of 200 pounds at a horizontal distance of 12 inches from the wheel and 21 inches from the end of the handles. What part of the load comes on the wheel and what part is lifted by the man?

78. Two men carry a load by means of a light bar, one man being at each end. If one man bears 80 pounds and the other 40 pounds, find the load and where it is placed on the bar.

79. Three men carry a load by means of a light bar, two men being at one end and one man at the other end. If each man lifts 35 pounds, find the load and where it is placed on the bar.

80. Change Fig. 32 so that the beam projects 5 feet beyond the right support, and at that end let there be an additional load of 203 pounds. Compute the upward pressures P_1 and P_2.

ART. 9. THE RESULTANT

The 'resultant' of a system of concurrent forces is found by the methods of Art. 4 in magnitude and direction. This resultant has the same effect as the given forces not only in regard to translation, but also in regard to rotation. Hence, referring to Fig. 34, if i be any axis normal to the plane of the forces P_1 and P_2, and l_1 and l_2 be their lever arms with respect to this axis, the tendency of the forces to cause rotation is expressed by the sum of the moments, or $P_1l_1+P_2l_2$. Let R be the resultant and c its lever arm, then its tendency to cause rotation is expressed by the moment Rc. Hence

$$Rc=P_1l_1+P_2l_2$$

and this may be easily extended to include any number of forces. Accordingly the following important principle may be stated:

> The moment of the resultant of several forces in the same plane is equal to the algebraic sum of the moments of the forces about any axis normal to the plane.

This formula and principle is also readily deduced from Fig. 31 of Art. 8, since P_3 if reversed in direction is the resultant of P_1 and P_2.

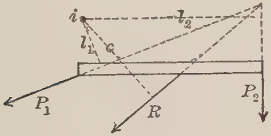

FIG. 34

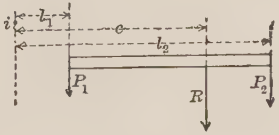

FIG. 35

Parallel forces acting on a body are really concurrent, since parallel lines meet at infinity. Hence all the methods of Art. 4 directly apply. If the forces are drawn vertically, ΣH becomes o, ΣV becomes ΣP, and the value of R is ΣP, while its direction is given by $\tan a=\pm\infty$, that is, it acts parallel to the given forces. The above principle of moments also directly applies. Thus, if i be any axis normal to the plane of the parallel forces in Fig. 35, the above equation gives the relation between the moment of the resultant and the moments of the forces.

The 'center line of a system of parallel forces' is the line in which their resultant acts, and its distance c from any assumed axis is given by the above equation for the case of two forces. If there are several forces P_1, P_2, P_3, etc., their moments are P_1l_1, P_2l_2, P_3l_3. The resultant is the algebraic sum of the forces ΣP, and the moment of the resultant is the algebraic sum of the moments of the forces. Hence the two formulas

$$R = \Sigma P \qquad\qquad Rc = \Sigma Pl$$

express the rules for finding the resultant and the center line of any system of parallel forces. In applying these rules the same conventions regarding signs of forces and signs of moments are to be used as in Art. 8.

In finding the center line the axis of moments may be taken at any convenient point; if it is taken on the line of action of one of the given forces, that force has no lever arm, its moment is 0, and hence the numerical work is simplified. For instance, in Fig. 36 the resultant R is $15+20-8-10=+17$ pounds and it acts upward. Taking the axis underneath the left-hand force, the equation of moments is $-17c=8\times3+10\times9-20\times7$, from which $c=+1.53$ feet, which shows that the center line is 1.53 feet to the right of the assumed axis. If the axis is taken at the left end of the beam, the moment equation is $-17c=8\times5+10\times11-15\times2-20\times9$, from which $c=+3.53$ feet, which gives the same position for the center line as before.

In Fig. 37 eight parallel downward forces, each equal to 3 pounds, are shown equally spaced 2 inches apart. Here $R = -24$ pounds acting downward. Taking the axis under the left-hand force, the moment of that force is 0, the moment of the next $+6$, of the next $+12$, and so on; thus $24c = 6 + 12 + \ldots + 42$, whence $c = +7$ inches, which gives the position of the center line.

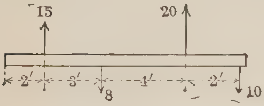

FIG. 36

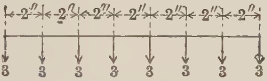

FIG. 37

PROBLEMS

81. Draw Fig. 36 so that the upward forces are 10 and 18 pounds and the downward forces 9 and 5 pounds. Find the resultant and the center line.

82. In Fig. 33 let $P_1 = 60$ pounds and $l_1 = 10$ feet. Find the resultant and the center line of the four forces.

83. In Fig. 35 take an axis at the distance d to the left of the one shown. Write the moment equation in terms of c, d, l_1, l_2, and show that it can be reduced to $Rc = P_1l_1 + P_2l_2$.

84. Two parallel forces are $+18$ and -10 pounds, their distance apart being 23 inches. Find the position of the center line.

85. Two parallel and equal forces act in opposite directions, their distance apart being 5 inches. What is the value of R, and what kind of motion will result?

86. A wheelbarrow carries a load P. What is the downward pressure on the wheel when a man lifts the handles with a force $\frac{1}{4}P$?

87. In Fig. 35 let $P_1 = 60$, $P_2 = 70$ pounds, and $l_2 - l_1 = 6$ feet. Find the resultant and its position.

88. Four parallel forces $P_1=+10$, $P_2=+16$, $P_3=-18$, $P_4=-20$ pounds are spaced equally distant. Find their resultant and center line.

89. The side of a car is 30 feet long and 8 feet high, and the wind blows against it with a pressure of 25 pounds per square foot. Where are the center lines of the wind pressure, and what is the total pressure?

90. Prove that it is impossible for three forces to be in equilibrium unless their lines of direction meet in a common point. Is it possible for four forces?

ART. 10. COUPLES

A 'couple' is a system consisting of two equal and opposite parallel forces not acting in the same straight line. Thus, in Fig. 38, there is shown a couple acting on the opposite sides of a square, and it is plain that this couple will cause the square to rotate in the positive or clockwise way. In Fig. 39 the couple causes rotation in the negative way. In Fig. 40 there are two couples, one tending to cause positive and the other negative rotation; here equilibrium can be secured if the forces are of the proper magnitudes.

A couple has no resultant, for in both Figs. 38 and 39 it is seen that $R=P-P=0$. In Fig. 40 the algebraic sum of the horizontal forces is $P_1-P_1=0$, and the algebraic sum of the vertical forces is $P_2-P_2=0$; hence two couples acting on a body have no resultant. There is, therefore, no tendency to a motion of translation when couples act on a body, but if motion results, it is one of rotation.

The 'moment of a couple' is the magnitude of one of the equal forces multiplied by the shortest distance between them. To show this let d be the distance between the forces in Fig. 38, and let i be an axis of moments at any distance e to the right of the right-hand side of the square. The algebraic sum of the moments with respect to this axis is $P(d+e)-Pe$, which is equal to Pd. If the axis is taken between the forces, say at the middle of the square, the algebraic sum of the moments is $P\times\frac{1}{2}d+P\times\frac{1}{2}d$, which is also equal to Pd. If the same reasoning is applied to Fig. 39, the moment will be found equal to $-Pd$, the negative sign indicating negative rotation.

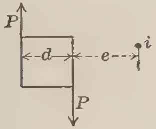

FIG. 38

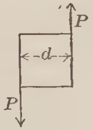

FIG. 39

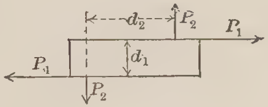

FIG. 40

When two couples act on a body tending to turn it in opposite ways, the body will remain at rest if the algebraic sum of the moments of the couples is zero, for then there is no tendency to rotation. Thus in Fig. 40 the algebraic sum of the moments about any axis is $P_1d_1-P_2d_2$; if the forces and distances have such values that this is zero, then equilibrium prevails. In this case P_1/P_2 must equal d_2/d_1, or the forces of the couples are inversely as their lever arms. For instance, if P_1 is ten times P_2, then d_1 is one-tenth of d_2.

When a system of parallel forces is found to have no resultant, it does not follow that they are in equilibrium, for the given forces may reduce to a couple that causes rotation. For example, in Fig. 30 let $P_1=-8$, $P_2=-9$, $P_3=+24$, $P_4=-7$ pounds, while $l_1=6$, $l_2=3$, $l_4=6$

inches. Here $R = -24 + 24 = 0$, so that there is no tendency to a motion of translation, but the equation of moments with respect to the axis i is $-8 \times 6 - 9 \times 3 + 7 \times 6$, which equals -33, showing that the forces will cause negative rotation to the same extent as a couple whose moment is -33 pound-inches. Such a couple might consist of forces of 3 pounds which are 11 inches apart, or forces of 6 pounds which are $5\frac{1}{2}$ inches apart, or of any other forces such that the product of one of them and the distance between them is -33 pound-inches.

Three forces having lines of action not meeting at one point cannot be in equilibrium, for they may be reduced to a couple, even when the algebraic sum of both the horizontal and vertical components is zero. To show this, draw a figure in which the force P_1 is taken as horizontal and let P_2 and P_3 intersect at A. If $\Sigma H = 0$, the algebraic sum of the horizontal components of P_2 and P_3 is a horizontal force acting at A which is equal and opposite to the force P_1, but as these are not in the same straight line, a motion of rotation ensues. The moment of this couple is equal to P_1 multiplied by its lever arm with respect to A.

PROBLEMS

91. In applying the formula $Rc = \Sigma Pl$ to the case of Fig. 36 what value is found for c? Why?

92. Which way does the body in Fig. 40 turn when $P_1 = P_2 = 13\frac{1}{2}$ pounds, $d_1 = 2$ inches and $d_2 = 1\frac{1}{2}$ inches?

93. When a force is applied to the handle of a grindstone, where is the other force of the couple?

94. Why is a screw-driver with a long handle easier to use than one with a short handle?

95. A horizontal force of 18 pounds acts eastward and one of 16 pounds acts westward on a body. What kind of motion does the body have if the forces act in the same line?

96. What kind of motion does the same body have if the distance between the lines of action of the two forces is 3 inches?

97. Draw a triangle whose sides are 3, 4, and 5 inches. Let forces of 6 and 8 pounds act along the first and second sides tending to cause positive rotation, and a force of 10 pounds along the third side tending to cause negative rotation. Do these forces cause a motion of translation?

98. If an axis be put at the middle of the shortest side of this triangle, in which way will rotation occur around it?

99. Draw three forces acting upon a body and not meeting at the same point. What kind of motion will result?

100. Explain the forces which cause the rotation of a wagon wheel when a horizontal pull is exerted upon the wagon.

ART. 11. NON-CONCURRENT FORCES

When several forces in the same plane act upon a body, they are most conveniently discussed by resolving each force into its horizontal and vertical components, thus reducing them to two systems of parallel forces at right angles to each other. For example, let Fig. 41 represent four forces applied at the corners of a rectangle; resolving these forces into their horizontal and vertical components, the two systems of parallel forces in Fig. 42 are found, and these must produce the same effect upon the body as the given inclined forces.

If the given forces do not meet at the same point, they are called 'non-concurrent,' and the conditions of equilibrium stated in Art. 5 are not sufficient, since they might be fulfilled and rotation still result. If H and V represent the horizontal and vertical components of a force, the conditions $\Sigma H=0$ and $\Sigma V=0$ state that no motion of translation occurs. Let P be any force and l its lever arm with respect to any point i in the plane; then $\Sigma Pl=0$ expresses the condition that there shall be no rotation (Art. 8). Accordingly

$$\Sigma H=0, \qquad \Sigma V=0, \qquad \Sigma Pl=0$$

are the conditions of equilibrium for non-concurrent forces in the same plane.

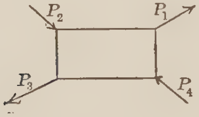

FIG. 41

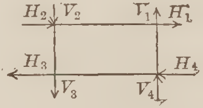

FIG. 42

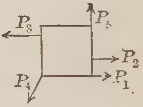

FIG. 43

After the given non-concurrent forces have been resolved into their components, as in Fig. 42, the condition ΣPl can be replaced by the condition that the algebraic sum of the moments of the components shall be zero. Let h and v be the lever arms of H and V with respect to any axis i; then ΣPl must be the same as $\Sigma Hh+\Sigma Vv$, since the components produce the same effect as the forces. Hence

$$\Sigma H=0, \qquad \Sigma V=0, \qquad \Sigma Hh+\Sigma Vv=0$$

are also the conditions of equilibrium for the two sets of components. Here ΣH means the algebraic sum of the horizontal components of the given forces, ΣV the algebraic sum of the vertical components, while ΣHh and ΣVv are the algebraic sums of the moments of the horizontal and vertical components respectively, these being taken about any axis normal to the plane.

For a system of concurrent forces it was shown in Art. 8 that the third of the above conditions is satisfied if the first and second are satisfied. For a system of non-concurrent forces, however, it may happen that the first and second conditions are fulfilled while the third is not; in this case rotation of the body will occur. If the second condition is satisfied and the others not, the body will have a horizontal motion of translation and also one of rotation.

These conditions may be used to determine whether a given system of forces is in equilibrium or not. If the system is known to be in equilibrium, they may be used to find unknown forces, directions, or lever arms, provided the number of such unknown quantities does not exceed three. In these computations it is always most convenient to use the horizontal and vertical components of the forces instead of the inclined forces themselves, that is, to consider the given forces as replaced by two systems of parallel forces.

For example, consider the two systems of parallel forces acting at the four corners of the square in Fig. 44. The algebraic sum of the longitudinal forces is zero and hence there is no horizontal motion. The sum of the vertical forces is zero and hence there is no vertical motion. Taking an axis of moments at the upper right-hand corner, the equation of moments is $-8d-12d=0$, where d is the side of the square, and hence there exists the moment $-20d$ which will cause the square to revolve in the opposite direction to that of the hands of a watch.

As a second example let a plate 8 inches long and 3 inches wide be placed in a vertical plane, as in Fig. 45, and be subject to the known horizontal force P_1, the known vertical forces P_2, P_3, and the unknown forces S, T, U, ∓ acting in the directions and positions shown. The first and second conditions of equilibrium give

$$P_1+T-S=0 \qquad\qquad P_2-P_3+U=0$$

while the third, taking the axis at the lower left-hand corner, gives

$$P_1 \cdot 1\tfrac{1}{2}+P_2 \cdot 0+P_3 \cdot 4-S \cdot 3-U \cdot 8+T \cdot 0=0$$

from which are found the values

$$S=\tfrac{1}{2}P_1+\tfrac{8}{3}P_2-\tfrac{4}{3}P_3, \quad T=-\tfrac{1}{2}P_1+\tfrac{8}{3}P_2-\tfrac{4}{3}P_3, \quad U=P_3-P_2$$

Thus if $P_1=6$, $P_2=9$, $P_3=12$ pounds, the values of the other forces must be $S=11$, $T=5$, $U=3$ pounds in order to insure equilibrium.

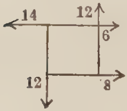

FIG. 44

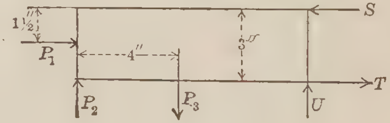

FIG. 45

In the discussion of cases of equilibrium it is immaterial where the axis of moments be taken, for there is no tendency to rotation about "any point" of the plane. Thus, in the last problem, if the axis be taken at the lower right-hand corner, the third condition is

$$P_1 \cdot 1\tfrac{1}{2}+P_2 \cdot 8-P_3 \cdot 4-S \cdot 3+T \cdot 0+U \cdot 0=0$$

from which the above value of S can be directly obtained. In like manner, if the axis be taken at the upper right-hand corner,

$$-P_1 . 1\tfrac{1}{2} + P_2 . 8 - P_3 . 4 + S . 0 - T . 3 + U . 0 = 0$$

from which T is directly found. By the successive application of the condition $\Sigma Pl = 0$, it is often possible to find all the unknown forces without using the first and second conditions, but these must also be satisfied in all cases of equilibrium, and hence checks upon the computation can be had.

PROBLEMS

101. Write the first and second conditions of equilibrium for the five forces in Fig. 43, where P_4 makes an angle of 30° with the vertical.

102. For the last problem let $P_5 = 100$ pounds. Find the value of P_4 so that there shall be no motion in a vertical direction.

103. Using the above values and letting $P_3 = 65$ and $P_2 = 50$ pounds, find the value of P_1 so that there shall be no motion in a horizontal direction.

104. Using the above values of the forces, let P_2 be applied at 2 inches above the base of the square and P_3 at the distance x below the top. Taking the side of the square as 12 inches, compute the value of x which is necessary to prevent rotation.

105. Draw a triangle whose sides are 3, 4, and 5 inches, and let a force act at the middle of each side and normal to it. Show that, for equilibrium, these forces must be proportional to the numbers 3, 4, 5.

106. If three non-concurrent forces act upon a body, show, by taking an axis of moments at the intersection of two of the forces, that rotation will result.

107. Explain how rotation in the last problem can be prevented by introducing a fourth force.

108, 109, 110. Three problems taken from other books may here be introduced for discussion and solution.

ART. 12. PARALLEL FORCES IN SPACE

In the preceding articles parallel forces in one plane have been considered, and the important case of parallel forces in space will now be discussed. Fig. 46 represents a simple instance where four parallel downward forces act at the four corners of a square which is in a horizontal position. If these four forces are to be held in equilibrium by a single upward force R, its value must be $P_1+P_2+P_3+P_4$. In general, also, the force which holds in equilibrium any number of parallel forces in space must be equal to their algebraic sum, and this force when reversed in direction is the resultant of the parallel forces.

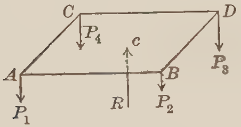

FIG. 46

FIG. 47

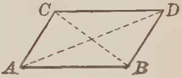

FIG. 48

In order to find the line of action of this resultant, a plane is to be passed normal to the given parallel forces, and two axes be drawn in this plane at right angles to each other. In the case of Fig. 46 the two lines AB and AC at right angles to each other are chosen as axes. If the five forces shown are in equilibrium, they have no tendency to turn around the axis AB and also no tendency to turn around the axis AC. The sum of the moments of the forces with respect to each of these axes must hence be equal to zero. Let l be the side of the square, c_1 the shortest distance from R to AB, and c_2 the shortest distance of R from AC. The equation of moments with respect to the axis AB is then $P_3l+P_4l-Rc_1=0$, the moments of P_1 and P_2 being zero because they lie upon AB. Similarly the equation of moments with respect to the axis AC is $P_2l+P_3l-Rc_2=0$. These two equations determine the position of the force R with respect to the axes of reference AB and AC. For example, let $P_1=6$, $P_2=7$, $P_3=10$, $P_4=11$ pounds, and the side of the square be 12 inches. Then $R=34$ pounds, and its line of action passes through the point c, which is 7.41 inches from AB and 6.00 inches from AC.

When forces in one plane are discussed their tendency to rotate about a point need alone be considered, but parallel forces in space require the consideration of the tendency to rotate about two axes. The moment of a force with respect to an axis lying in a plane normal to the force is the product of the force by the shortest distance from the axis. The conditions of equilibrium for parallel forces in space are hence three in number: first, that the algebraic sum of all the forces shall equal zero; second, that the sum of the moments of the forces with respect to an axis in a plane normal to the forces shall be zero; third, that the sum of the moments of the forces with respect to a second axis in the same plane and at right angles to the first one shall also be zero.

PROBLEMS

111. Fig. 47 represents a triangular table with a leg at each corner. If a load P is placed at the center of gravity of the table, what part of P is borne by each leg?

112. In Fig. 46 let $P_1=P_2=P_3=6$ pounds, and $P_4=0$. Find the magnitude and position of the resultant.

113. In Fig. 46 let P_1, P_2, P_3 be each equal to 6 pounds, and P_4 be 8 pounds acting upward. Find the magnitude and position of the resultant.

114. Let Fig. 48 represent a thin sheet of tin in the shape of a parallelogram. If it be in a horizontal position, show that there is no tendency to turn about the diagonal AD.

115. If a single upward force be applied to balance this tin parallelogram, show that it must be put at the intersection of the two diagonals.

116. At the three vertices of a horizontal equilateral triangle there act forces of 10, 10, and 5 pounds. Find the position of the force that will hold them in equilibrium.

117. Solve the last problem, taking the three forces as 5, 5, and 10 pounds; also as 7, 8, and 9 pounds.

118. If a horizontal wind pressure of 35 pounds per square foot acts against the side of a car 8 feet high and 30 feet long, what is the magnitude and position of the resultant?

119. A horizontal triangle ABC has $AB = 12$ feet, $AC = 5$ feet, and $BC = 13$ feet. If three equal forces are placed at the three vertices, find the distances of their resultant from AB and AC.

120. In the same triangle, let AD be a line drawn from A to the middle of the opposite side, and let E be a point on this line such that DE is one-third of AD. If equal loads be placed at the three vertices, show that their resultant passes through E.

CHAPTER III

CENTER OF GRAVITY

Art. 13. Definitions and Principles

The force of gravity acts downward upon all bodies at the surface of the earth and, since the earth is very large compared with the bodies to be considered, the lines of action of the forces exerted by gravity upon all the particles of a given body are sensibly parallel. The resultant of all these forces of gravity is equal to the weight of the body, and the line along which this resultant acts passes through a point called the 'center of gravity of the body.' For example, if a coin be balanced upon its edge, the resultant of all the forces of gravity passes through the axis of the cylinder; if it be held in a horizontal position, the resultant coincides with that axis. The center of gravity of a coin hence lies half-way between the center of its two circular faces.

Since the resultant of any number of forces may be held in equilibrium by an equal and opposite force, it follows that a body may be balanced upon a point placed vertically above or below it. For instance, if a thin sheet of cardboard or metal be held in a horizontal position, a point may be found by trial upon which it may be balanced by a pin there placed, and the center of gravity of the sheet is just above the point of the pin.

The following definition of the center of gravity will generally be used in computations:

> The center of gravity of a body or system of bodies is a point where the resultant weight may be concentrated and have the same tendency to rotate about any axis as the actual distributed weight.

For example, take the line of eighteen equal particles spaced at equal distances apart in Fig. 49. In order to find the center of gravity c from the above definition, let w be the weight of each particle and d the distance between the centers of two consecutive ones. The resultant being $18w$ and the total length $17d$, the principle of moments gives, for an axis at A, the equation

$$18w \times AC = w \times d + w \times 2d + w \times 3d + w \times 4d + \dots + w \times 17d$$

from which AC equals $8\frac{1}{2}d$, that is, the center of gravity of the particles is at the middle of the line.

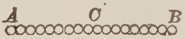

FIG. 49

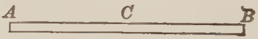

FIG. 50

The above definition may also be stated in a different way, namely: The center of gravity of a body or bodies is a point such that there is no tendency to rotation about any axis drawn through it. Thus, for the bar in Fig. 50, the downward forces of gravity will cause rotation about any axis except one passing through the middle, since the moment of the weight on one side will be greater than the moment of the weight on the other side. The center of gravity of any horizontal bar of uniform cross-section and material hence lies at its middle point.

A line, or an area, or a geometric volume has no weight and hence no center of gravity, but it is customary to consider them as having weights proportional to their magnitudes. A line may represent a bar of uniform size, an area a sheet of metal, and a volume a solid body. In speaking of the centers of gravity of geometric figures it is hence understood that they represent the centers of gravity of matter uniformly distributed along the line, over the area, or throughout the volume.

The last definition above given shows that, when a straight line can be drawn so as to symmetrically divide a geometric figure, its center of gravity lies on that line, since the action of parallel forces can have no tendency to cause rotation about such an axis. Thus the center of gravity of an arc of a circle lies on a radius bisecting it, the center of gravity of a square lies on its diagonal, and the center of gravity of a right cone lies on the line joining the vertex with the center of the base. If a geometric figure has two lines of symmetry, the center of gravity must be at their intersection; thus the center of gravity of a rectangle is at its middle point, and the center of gravity of a sphere at its geometric center.

The terms 'center of mass' and 'centroid' are frequently used instead of center of gravity, because this point is frequently to be considered in other connections than that of weight. In all cases, however, this important point is on the line of action of the resultant of parallel forces. The earth and the moon constitute a system of bodies which has, in the usual sense, no weight, but it has a centroid.

PROBLEMS

121. Where is the center of gravity of the circumference of a circle? of the area of a circle?

122. Three equal spheres lie upon the same straight line, the distance between the centers of the first and the second

being 35 feet and that between the centers of the second and third being 7 feet. Find their center of gravity by the method of Art. 7.

123. The rectangular table $ABCD$ has $AB=4$ feet and $BC=3$ feet. At A, B, C, D are bodies weighing 3, 2, 4, and 12 pounds. Find their center of gravity by the method of Art. 12.

124. Solve the last problem, taking the weight at D as 150 pounds, the other data remaining the same.

125. Draw a circular sector and a line upon which its center of gravity lies.

126. Draw an equilateral triangle and two lines upon which its center of gravity lies.

127. For Fig. 50 find the center of gravity, if the half-length AC weighs 20 pounds, while BC weighs 15 pounds.

128. Where is the center of gravity of a cube? a parallelopiped? a wedge? a pyramid?

129. A circle of 10 inches radius has a circular hole of 3 inches radius, the center of the hole being 5 inches from the center of the circle. Draw a line upon which the center of gravity of the area lies.

130. The mean distance between the centers of the moon and earth is 240 000 miles, and the quantity of matter in the earth is 82 times that in the moon. How far from the center of the earth is the centroid of the two bodies?

ART. 14. CENTERS OF GRAVITY FOR LINES

The word 'line' here means a collection of small material particles uniformly distributed along the geometric line. Straight wires of metal and rods of metal or timber may be regarded, for the purpose of finding their centers of gravity, as straight lines coinciding with their geometric axes. The center of gravity of a straight rod of uniform size and material is at the middle of a straight line joining the centers of its ends (Art. 13).

A compound rod, formed of two straight rods of different sizes or of different materials, has its center of gravity between the centers of gravity of the two parts. Let Fig. 51 represent two straight rods joined together, the weight of the first being W_1 and that of the second W_2. These two weights can be regarded as concentrated at the centers of gravity of the two parts. The center of gravity of the compound rod lies upon the center line of the two parallel forces W_1 and W_2. Let l be the distance between these forces, and x the distance of W_1 from this center line; then, taking an axis on this center line, the principle of moments gives the equation $W_2(l-x)-W_1x=0$, and the value of x which satisfies this equation locates the resultant R of W_1 and W_2, and this passes through the center of gravity of the compound rod. For example, let $W_1=10$, $W_2=5$ pounds, and $l=36$ inches, then $x=12$ inches, and an upward force of 15 pounds placed at 12 inches to the right of the middle of the larger section will balance the compound rod.

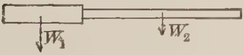

FIG. 51

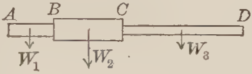

FIG. 52

Fig. 52 shows a compound rod formed of three straight rods of different sizes. Let the lengths AB, BC, CD be 2, 3, 5 feet and the weights of the same be 12, 43, 18 pounds respectively. The total weight is 73 pounds, and this resultant acts in a vertical line passing through the required center of gravity. Let x be the distance of this line from the end A; then, by the principle of moments,

$$73x=12\times 1+43\times 3\tfrac{1}{2}+18\times 7\tfrac{1}{2}$$

from which $x=4.07$ feet, so that the center of gravity lies 0.57 feet to the right of the middle of BC.

A curved line or wire ABC which is symmetrical with respect to a straight line BO, so that every chord BC which is normal to BO is bisected, is shown in Fig. 53. The center of gravity of the curved line ABC lies upon the line BO (Art. 13), but its position upon that line will depend upon the nature of the curve. For an arc of a circle whose center is at O, it may be shown, by the methods of calculus, that the distance of the center of gravity from O is equal to the radius BO multiplied by the chord AC and divided by the arc ABC.

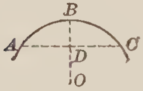

FIG. 53

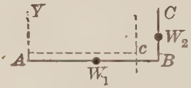

FIG. 54

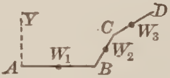

FIG. 55

A bent line or rod ABC composed of two straight parts AB and BC at right angles to each other is shown in Fig. 54. The center of gravity of AB is at its middle, and its weight W_1 may be taken as there concentrated. In like manner the weight W_2 of BC may be taken at the middle of BC. The center of gravity of the two rods is then at the same point as the center of gravity of the weights W_1 and W_2. Let AB and BC be in a horizontal plane, then these weights act downward and tend to cause rotation about an axis AY drawn at A normal to AB.

Let l_1 and l_2 be the lengths of AB and BC; then the lever arms of W_1 and W_2 with respect to the axis AY are $\frac{1}{2}l_1$ and l_1. Let x be the distance of the center of gravity c from AY. Then the principle of moments gives the equation $(W_1+W_2)x=W_1\times\frac{1}{2}l_1+W_2l_1$. The weights also tend to cause rotation around AB as an axis, and if y be the distance of c from this axis, the equation of moments is $(W_1+W_2)y=W_2\times\frac{1}{2}l_2$; the weight W_1 does not enter this equation, since its lever arm is zero. The values of x and y found from these two equations locate the position of the center of gravity of AB and BC. If w be the weight of the rod per linear unit, W_1 may be replaced by wl_1 and W_2 by wl_2, and the value of x then reduces to $(\frac{1}{2}l_1^2+l_1l_2)/(l_1+l_2)$, while that of y reduces to $\frac{1}{2}l_2^2/(l_1+l_2)$. When the rods are of the same length l, the value of x is $\frac{3}{4}l$, while that of y is $\frac{1}{4}l$.

By similar reasoning the position of the center of gravity of the bent line $ABCD$ in Fig. 55 may be ascertained. As a special case let AB be 3 feet, BC be $1\frac{1}{2}$ feet, and CD be 2 feet, BC making an angle of 60° and CD one of 30° with a line parallel to AB. The weights of the parts AB, BC, and CD may be taken as 3, 1.5, and 2, these numbers being proportional to their lengths; and their lever arms with respect to AY are 1.5, 3.37, and 4.61 feet. Let x be the distance of the center of gravity of these weights from the axis AY; then $6.5x=3\times1.5+1.5\times3.37+2\times4.61$, whence $x=2.89$ feet. In the same way by taking an axis coinciding with AB the distance from it to the center of gravity is found to be 0.70 feet. These values of x and y locate the center of gravity of the bent line $ABCD$.

PROBLEMS

131. The large rod in Fig. 51 is 3 feet long and weighs 7.5 pounds per linear foot, while the small one is 5 feet long and weighs 2.0 pounds per linear foot. Find the center of gravity.

132. The lengths of AB, BC, and CD in Fig. 52 are 2.0, 1.8, and 3.0 feet, and their weights are 6, 12, and 5 ounces. Find the center of gravity.

133. From the above statement regarding the center of gravity of a circular arc deduce the position of the center of gravity of a semicircle.

134. Where is the center of gravity of a straight steel pipe of uniform cross-section?

135. A bent rod is formed of two straight rods of equal size at right angles to each other. Find the center of gravity when one rod is 12 and the other 3 feet long.

136. In Fig. 55 let the lengths of AB, BC, CD be 4, 3, 2 feet, BC making an angle of 135° with AB, and CD being parallel to AB. Find the values of x and y which locate the center of gravity.

137. A straight rod 6 feet long weighs 10 pounds and has a weight of 12 pounds hung at one end. Find the center of gravity of the two bodies.

138. A straight rod 6 feet long weighs 10 pounds and has a weight of P pounds hung at one end of it. What is the value of P if the center of gravity of the two bodies is 2 feet distant from P?

139. Show that the center of gravity of the three sides of a triangle is at the same point as the center of gravity of three weights placed at the middle points of the sides, these weights being proportional to the lengths of the sides.

140. Find the position of the center of gravity of the three sides of a triangle, their lengths being 5, 12, and 13 inches.

ART. 15. CENTERS OF GRAVITY FOR SURFACES

A plane surface has no weight and hence no center of gravity, but it is customary to speak of geometric figures as if they had weights proportional to their areas, they being regarded as representing actual sheets or plates of matter. When such a surface is held in a horizontal position, a point may be found upon which it may be balanced by a pin there placed, and this point is the center of gravity of the surface. The center of gravity of a circle, a rectangle, or a parallelogram is at its geometric center; for, if any line be drawn through this point, the weight upon one side is equal to that upon the other side, and the moment of the weight on one side with respect to that line is equal to that for the other side. Thus, in Fig. 56, the center of gravity of the rectangle lies upon the diagonal shown, and it also lies upon the other diagonal; accordingly the center of gravity of the rectangle is at the intersection of the two diagonals. The center of gravity of the triangle in Fig. 57 is at the intersection of the median lines Aa and Bb, for Aa bisects any line parallel to BC and hence the center of gravity must lie upon it, and for a similar reason it must lie upon Bb.

When a thin sheet of cardboard or metal is hung in a vertical position upon a horizontal pin, the center of gravity of the sheet must lie vertically below the pin, for rotation will occur around the pin if the resultant of the forces of gravity does not pass through it. The following is hence a practical method for locating the center of gravity of any plane area. Let a sheet of metal or cardboard of uniform thickness be cut to the required shape and be suspended upon a pin put through it near one edge. While in this position let a vertical line be drawn upon the sheet by the help of a plumb-bob hung by a string from the pin. Then let the sheet be hung again with the pin put in another position and a second vertical line be drawn upon it. The intersection of these two lines is the center of gravity of the plane area. The student should try this experiment, locating the second position of the pin so that the two lines may be approximately at right angles, in order that a definite point of intersection may be obtained.

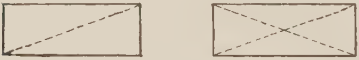
FIG. 56

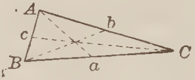

FIG. 57

In many simple cases the center of gravity may be ascertained from computations by the help of the principle of moments. For example, the plane area $ABCDEF$ in Fig. 58 may be regarded as composed of two rectangles having the weights W_1 and W_2, these being proportional to the areas of $ABCG$ and $DEFG$. If AB be represented by a, BC by b, DE by c, and EF by d, W_1 may be taken as the area ab and W_2 as the area cd, and their resultant R by $ab+cd$. Taking moments with respect to the axis AB, the lever arm of W_1 is $\frac{1}{2}b$, that of W_2 is $b+\frac{1}{2}c$, that of R is x, and the equation of moments is

$$(ab+cd)x = ab\times\tfrac{1}{2}b + cd(b+\tfrac{1}{2}c)$$

and the value of x found from this equation locates a line parallel to AB upon which the center of gravity lies. Again, let the line AF be taken as an axis of moments;

then the equation of moments is

$$(ab+cd)y=ab\times\tfrac{1}{2}a+cd\times\tfrac{1}{2}d$$

and the value of y locates a line parallel to AF upon which the center of gravity lies.

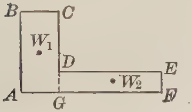

FIG. 58

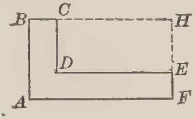

FIG. 59

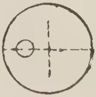

FIG. 60

Another way of finding the center of gravity of this area may be illustrated by Fig. 59, where $ABCDEFA$ is regarded as formed by taking the rectangle $CDEH$ away from the rectangle $ABHF$. Let AB and AF be 3 and 5 inches respectively; then $ABHF$ is 15 square inches, and its lever arms with respect to the axes AB and AF are $2\frac{1}{2}$ and $1\frac{1}{2}$ inches. Let CD and DE be 2 and 4 inches; then $CDEH$ is 8 square inches, and its lever arms with respect to the axes AB and AF are 3 and 2 inches. Let x and y be the lever arms of $ABCDEFA$ with respect to AB and AF; then from Art. 12,

$$(15-8)x=15\times2\tfrac{1}{2}-8\times3 \qquad (15-8)y=15\times\tfrac{1}{2}-8\times2$$

from which $x=1.93$ and $y=0.93$ inches, and these values locate the center of gravity.

Fig. 60 shows a circular plate of diameter D through which a hole of diameter d is cut, the center of the hole being at the distance h from the center of the circle. The center of gravity of the area lies upon a line drawn through these two centers, since this is a line of symmetry (Art. 13). To find its position on this line, an axis may be taken at any point and the equation of moments be stated, the area and the moment of the hole being taken as negative. The distance of the center of gravity from the center of the circle will be found to be $hd^2/(D^2-d^2)$.

The center of gravity of a curved surface usually lies without the surface. Thus, if the curved line *ABC* in Fig. 53 be revolved about the axis *BO*, it generates a surface of revolution which is symmetrical with respect to the line *BO*, and its center of gravity hence lies upon this line. If *ABC* be an arc of a parabola, the surface generated is that of a paraboloid, and it can be shown by the differential calculus that the distance from *B* to the center of gravity of this surface is $\frac{5}{8}\overline{BD}$.

PROBLEMS

141. Find the center of gravity of a parallelogram.

142. Show that the center of gravity of a triangle lies upon a line drawn parallel to the base at a distance from the base equal to one-third of the altitude.

143. Make an experiment to find the center of gravity of an irregular sheet of cardboard by hanging it from a pin in three positions.

144. For Fig. 58 compute the position of the center of gravity when $AB=6$, $AF=10$, $BC=2$, and $EF=2$ inches.

145. For Fig. 59 compute the position of the center of gravity when $AB=6$, $AF=6$, $BC=2$, and $EF=2$ inches.

146. In a square whose side is 6 inches there is cut a circular hole of $2\frac{1}{2}$ inches diameter, the center of the hole being half-way between the center of the square and one of its corners. Find the center of gravity of the area.

147. If a trapezoid has the lower base b_1, the upper base b_2,

and the altitude h, show that its center of gravity lies at a distance $h(b_1+2b_2)/3(b_1+b_2)$ above the lower base.

148. Two coins, having diameters of 2 and 4 centimeters, lie upon a table, the distance between their centers being 3 centimeters. Find their common center of gravity.

149. Take a rectangular card and cut two holes in it near one side. Estimate the position of the center of gravity of the area, and test this estimate by computation or experiment.

150. A long rectangular plate has one-third its length of one kind of metal and the remainder of a different kind. What are the relative weights of the metals if the plate can be balanced at a point half-way between the middle and one end?

ART. 16. STABLE AND UNSTABLE EQUILIBRIUM

A body at rest is said to be in 'stable equilibrium' when it tends to return to its original position after having been deflected therefrom by a small horizontal force. For example, if a ball be suspended from a fixed pin by a string as in Fig. 61, a slight horizontal force applied to the ball will cause a deflection so that the string is no longer vertical, but on the removal of the force the ball returns to its original position, which is one of stable equilibrium. In the position of rest the ball is held in equilibrium by two forces, the downward weight of the ball and the upward pressure of the pin, and these forces are in the same vertical line; after the lateral deflection the two forces form a couple (Art. 10) which tends to rotate the ball around the fixed pin to its former position. Similarly in Fig. 62 the ball resting at the lowest part of the curve is in stable equilibrium, for if deflected laterally it tends to roll back.

Since the entire weight of a body concentrated at its center of gravity produces the same tendency to rotation as the actual distributed weight (Art. 13), the conditions of stable equilibrium are, first, that the center of gravity of the body shall lie in the same vertical line as the point of support, and, second, that after the lateral deflection this weight and the pressure of the support shall form a couple which tends to restore the body to its original position.

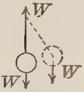

FIG. 61

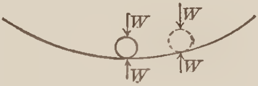

FIG. 62

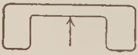

FIG. 63

A body at rest is said to be in 'unstable equilibrium' when it tends to move away from its original position after having been deflected therefrom by a small horizontal force. For example, the balls in Figs. 64 and 65 tend to rotate away from their original positions, if deflected laterally by a small force, since a couple is formed by the downward weight of the ball and the upward pressure of the support, and this causes such rotation. The conditions of unstable equilibrium hence are that the center of gravity of the body shall lie on the same vertical line as the point of support, and that after the lateral deflection a couple is formed which rotates the body away from its original position.

In stable equilibrium the center of gravity of the body rises above its original position under the action of a lateral force, and in unstable equilibrium it falls. Hence the conditions of these two kinds of equilibrium may be expressed more simply by saying that if any lateral force causes the center of gravity of the body to rise, the equilibrium is stable; while if it causes the center of gravity to fall, the equilibrium is unstable. Fig. 63 represents a body where the center of gravity rises when a slight horizontal force causes rotation around the point of support, and this is a case of stable equilibrium. Fig. 66 represents a body where the center of gravity falls when a slight rotation is caused around the point of support, and this is a case of unstable equilibrium.

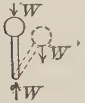

Fig. 64

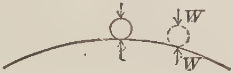

Fig. 65

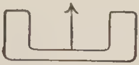

Fig. 66

There is a third kind of equilibrium called 'neutral equilibrium' in which the body, after having been deflected from its original position by a slight horizontal force, has no tendency either to return to or depart further from the same. For example, a sphere or a cylinder lying on a level table is in neutral equilibrium, for its center of gravity neither rises nor falls when it is moved laterally by a small force. An ellipsoid with three unequal axes placed on a level table is in stable equilibrium if its shortest axis is vertical, it is in unstable equilibrium if its longest axis is vertical, and it is also in unstable equilibrium if its third axis is vertical.

PROBLEMS

151. If the body in Fig. 61 weighs 10 pounds, show that a horizontal force of 2.68 pounds, acting through the center of gravity, is necessary in order to keep the string at an angle of 15 degrees with the vertical.

152. When the string in Fig. 61 is vertical and the body weighs 10 pounds, what is the tension in the string?

153. If the string in Fig. 61 is kept at an angle of 30 degrees with the vertical by a horizontal force, what is the tension in it when the body weighs 10 pounds?

154. Make an experiment to confirm Fig. 63 by using two pocket knives and a piece of wood.

155. A cone rests upon its side on a level table. In what kind of equilibrium is it? Why?

156. A vertical wheel can revolve upon a horizontal axis. In what kind of equilibrium is it? Why?

157. What is a position of a right cone so that its equilibrium may be unstable? What is a position for stable equilibrium?

158. The center of gravity of a hemisphere is at a distance of $\frac{3}{8}r$ from the center of its plane base r being the radius. If a cylinder of the same material and of the height h be placed upon this plane base and be joined to it, show that the center of gravity of the compound body lies at the center of that base when $2h^2$ equals r^2.

159. If the body of the last problem be placed with its spherical surface in contact with a level table, show that it will be in stable equilibrium when r^2 is greater than $2h^2$, and in unstable equilibrium when r^2 is less than $2h^2$.

160. A plank 3 feet long, 1 foot wide, 2 inches thick, and weighing 20 pounds rests upon a support at the middle, and a boy sits at each end. Show that the equilibrium is unstable. If the boys stand upon the ends, is the instability increased or decreased?

160*a*. Draw a curve having the right-hand part like that of Fig. 65 and the left-hand part like that of Fig. 62. Let a ball be placed on the curve at the point where it is horizontal. What is the nature of the equilibrium of the ball?

Art. 17. Stability against Rotation

When a body rests upon the surface of the earth, it has no motion either up or down. Since, however, the force of gravity tends to pull it downward, the earth must exert an equal upward force upon it (Art. 2, Axiom 3). If no horizontal force acts upon the body, the upward resultant pressure must pass through the center of gravity of the body, since the resultant downward force acts through that point. Thus in Fig. 67 the downward force W is the resultant of all the forces of gravity or the weight of the body, and the upward force W is the resultant of all the upward earth pressures. When a body has such a shape that the line of action of its weight does not pass through the base, as in Fig. 68, it cannot be supported on that base, since the two forces form a couple which causes rotation.

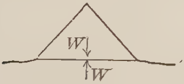

Fig. 67

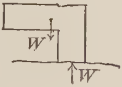

Fig. 68

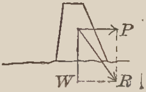

Fig. 69

When a body of weight W rests on a level base and is subjected also to a horizontal force, P, as in Fig. 69, the resultant R of these two forces must be counteracted by an equal and opposite force R acting in the same line in order that equilibrium may prevail. Such a reacting force cannot be exerted upon the body unless the line of action of R cuts its base. Rotation will hence occur if the resultant of all the forces acting on the body does not intersect the base of support. For example, the vertical line through the center of gravity of a man must cut the ground within the base formed by his shoes, or else he will fall; if he be subject to the action of a strong wind, he must lean to the windward in order to preserve his equilibrium.

This general principle may be illustrated by the following figures, in each of which a rod is seen resting upon a table and projecting over its edge. In Fig. 70 the rod is straight and its center of gravity is at its middle C; in Fig. 71 the rod is bent at right angles and the center of gravity is nearest to the vertical part; in Fig. 72 the rod is bent at an acute angle and a weight hung at its lowest end. In each case C represents the point where the vertical line through the center of gravity cuts the base AB; if this point lies beyond B, rotation will occur.

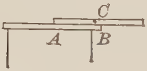

FIG. 70

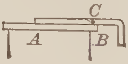

FIG. 71

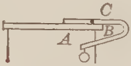

FIG. 72

The stability against rotation of a body resting on a horizontal plane is the greatest when the resultant of all the acting forces cuts the base at its middle. When this resultant cuts the edge of the base, the body is about to fall; when it passes without the base, failure by rotation is sure to occur.

The magnitude of a horizontal force required to overturn a body may be computed from the principle of moments. For example, let ABC in Fig. 73 represent the end of a triangular prism of weight W, the side AC being vertical. Let b be the width of the base AB, then the vertical line through the center of gravity of the prism cuts the base at a distance $\frac{1}{3}b$ from the edge A (Art. 15). Let P be the horizontal force required to overturn the prism, and h be its distance above the base. At the moment when rotation begins, the resultant R of W and P must pass through the edge B. Taking this point as an axis of moments, the equation of moments is $P\times h-W\times\frac{2}{3}b=0$, and hence $P=W.2b/3h$ is the horizontal force which causes rotation to begin; the moment of R does not enter this equation because its lever arm is zero.

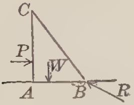

Fig. 73

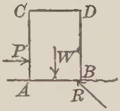

Fig. 74

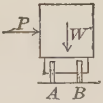

Fig. 75

As another example, let it be required to find what horizontal wind pressure per square foot will overturn a railroad car weighing 21 000 pounds, the end of which is shown in Fig. 75. The car body is 30 feet long, 8 feet high, 8 feet wide, and its floor is 3 feet above the rails, while the distance between the centers of the rails is 5 feet. The number of square feet on the side of the car is then 240, and, if w be the wind pressure per square foot, the force P is $240w$. When the resultant of the weight and the horizontal force $240w$ passes through B, rotation is about to begin; the lever arm of the weight with respect to B is $2\frac{1}{2}$ feet, while that of $240w$ is 7 feet, since the resultant of all the wind forces acts at the middle of the car. The equation of moments with respect to B then is $240w\times 7-21\,000\times 2\frac{1}{2}=0$, from which w is found to be 31 pounds per square foot nearly.

In order to increase the stability of a proposed structure against rotation its weight may be increased, or its shape may be altered so as to shift the center of gravity in a direction that will increase the resisting moment. Fig. 74 represents a dam of rectangular section subject to the horizontal water pressure P; the base may be made wider and the top narrower without changing the weight W, but the lever arm of W with respect to B is thus made greater and the stability of the structure against rotation is hence increased.

PROBLEMS

161. The entire length of the rod in Fig. 71 is 2 feet and the vertical part is 6 inches in length. Find the least length of the base AB.

162. Try the experiment indicated by Fig. 72.

163. In Fig. 73 let W be 100 pounds, P be 18 pounds, and AB be 3.5 feet. At what height above AB must P be applied to cause rotation?

164. Let Fig. 74 represent a cylinder standing on end, its height being 60 inches, the radius of its base 12 inches, and its weight 273 pounds. What force P applied at the middle of the height will cause rotation?

165. In Fig. 73 let AB be 3.5 feet and AC be 8.6 feet, the length of the prism normal to the plane of the paper being 10 feet, and the weight of the material being 140 pounds per cubic foot. What horizontal force P applied at the middle of the height will cause the resultant to cut the base at its middle?

166. A brick is laid with one-fourth of its length projecting over the edge of a table. Upon this is laid a second brick with one-fourth of its length projecting over the first. Upon this is laid a third brick with a similar projection. Are the three bricks stable, or will they fall?

167. Show that the stability against rotation of the body

in Fig. 74 is the greatest when P is zero, and that the stability of the body in Fig. 73 is the greatest when P equals $W \,.\, b/6h$.

168. In Fig. 74 let $W = 100$ pounds, $P = 30$ pounds acting at the middle of the height, $AB = 6$ feet and $AC = 18$ feet. Is the body stable against rotation?

169. For the last problem find where the resultant of the forces W and P cuts the base AB.

170. A table 3 feet square has its top projecting 3 inches beyond the four legs. If the table weighs 27 pounds, what weight hung from one of the corners will overturn it?

CHAPTER IV

RESISTANCE AND WORK

ART. 18. RESISTING FORCES

An 'applied force' is one exerted upon a body by an exterior force, like muscular effort, gravity, pressure, or pull. Such applied forces may cause the body to move or they may meet with resistance which prevents motion. A 'resisting force' is one that opposes the applied force and which disappears when the applied force is removed. For example, if a beam is placed upon a support, its weight acts downward and an upward resisting force called the 'reaction of the support' is generated; if a force is applied to slide a body which rests on a table, a resisting force called 'friction' comes into play; if a wagon is drawn, a force called 'traction resistance' opposes the motion; if a ball is thrown by the hand, a resisting force called 'inertia' is felt. When motion does not occur, the resisting forces are such as to hold the applied forces in equilibrium; thus, if a horse pulls against a fixed post, the post pulls the horse with an equal resisting force. The principles of the preceding chapters enable the resisting forces to be found for all cases of equilibrium.

Friction is one of the most common resisting forces, and this is known by experience to occur whenever a force tends to slide one body along another. Fig. 76 shows a body resting upon a horizontal plane, its weight W being held in equilibrium by the equal upward resisting pressure R. When a horizontal force P is applied to this body, as in Fig. 77, the horizontal resisting friction F comes into play between the surfaces of contact. If P is not large enough to cause motion, F must be equal to P (Art. 5), but they do not generally act in the same line, so that they form a couple which tends to cause rotation (Art. 10). Hence the point of application of the upward resisting pressure must shift so that the moment of W and R balances the moment of P and F.

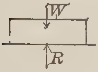

FIG. 76

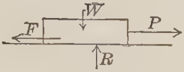

FIG. 77

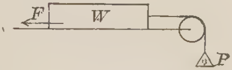

FIG. 78

If the horizontal force P is large enough, it will cause the body to slide, because the applied force overcomes the resisting friction. Friction is due to the roughness of the surfaces of contact, all surfaces being more or less rough, so that the projections of each surface fall into depressions upon the other surface. When a force is applied to cause sliding, these projections must be broken off or be lifted out of the depression before motion can occur. The smoother the surfaces the less is the resisting friction.

The resisting force of friction may be measured by the simple apparatus shown in Fig. 78; this consists of a pan at the end of a cord which passes over a pulley and is attached to the body W in such a way as to pull horizontally upon it when weights are placed in the pan. These weights are added in succession until finally the body begins to slide. At any moment before sliding the resisting friction F is equal to the weight P in the pan; at the instant of sliding this force F has its greatest value.

It has been found by such experiments that this force of friction is greater for rough surfaces than for smooth ones, and that it increases directly as the weight of the body which is moved.

Another method of measuring the force of friction is shown in Fig. 79, which represents a body of weight W about to slide upon an inclined plane AC. Let a be the angle which AC makes with the horizontal AB, and let there be a hinge at A so that this angle may be slowly increased until sliding occurs. Let W be resolved into the components P and N, the former acting parallel and the latter normal to AC. Since W is normal to AB, the angle between W and N is the same as the angle a, and N equals $W \cos a$, while P equals $W \sin a$. Now at the instant of sliding the resisting friction F equals the applied force P which is parallel to it; hence $F = W \sin a$, and thus the value of F is known by the measurement of W and a. For example, if the body is a brick which weighs 4.5 pounds and if this begins to slide when the angle a is 24 degrees, then the resisting friction is 1.83 pounds.

PROBLEMS

171. What is the resisting force which comes into action when a man pushes against a wall? When a weight is hung at the end of a vertical string?

172. A string 20 feet long is fastened to the tops of two posts of equal height 18 feet apart, and a weight of 4.16 pounds hung at the middle. Find the resisting forces in the string.

173. In Fig. 77 let P be 12.5 pounds applied at 3.25 inches above the base. Let the body be a parallelopiped weighing 25.6 pounds. What are the values of F and R and where are they applied?

174. Two bodies of equal roughness rest upon the same table, one weighing three times as much as the other. If a horizontal force of 1.25 pounds causes the heavier body to move, what force will cause the lighter one to begin moving?

175. A horizontal force of 6.5 pounds causes a body weighing 18 pounds to begin motion on a horizontal plane. If the same body rests upon a similar surface, what must be its inclination to the horizontal so that the body may begin to slide?

176. In Fig. 79 let AB be 6.3 and BC be 2.5 feet. Find the angle which AC makes with the horizontal.

177. In Fig. 79 let AC be $18\frac{1}{4}$ and BC be $6\frac{3}{8}$ inches. Find the sine of the angle BAC.

178. A body resting upon an inclined plane AC weighs 3 kilograms, AC being 12 and BC being $5\frac{3}{4}$ inches. Compute the normal pressure on the plane and the resisting friction.

179. What are two of the resistances which oppose the motion of a wheel when revolving on its axle? If a tangential force of 6 pounds at the circumference causes such a motion to begin, where is the resisting friction?

180. Draw a figure showing a body on an inclined plane subjected to a horizontal push P. Where and in what direction does the resisting friction act?

ART. 19. FRICTION

The resisting friction which acts between two surfaces when one just begins to slide upon the other has been often measured by the methods described in the last article, and from such experiments the following facts or laws have been ascertained:

1. The force of friction is directly proportional to the normal pressure between the surfaces.
2. It is independent of the area of contact.
3. It depends upon the nature of the surfaces, being smaller for smooth surfaces than for rough ones.

These laws of friction may be algebraically expressed by the simple formula

$$F = nN$$

in which N is the normal pressure between the two surfaces, F is the resisting friction at the moment when sliding begins, and n is an abstract number called the 'coefficient of friction,' which varies with the nature of the surfaces.

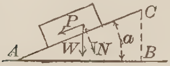

FIG. 79

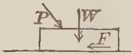

FIG. 80

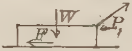

FIG. 81

The above laws are approximate ones, but they fairly represent the general facts of observation when the normal pressures are not excessive. If a body rests upon a horizontal plane, as in Fig. 78, the normal pressure between the surfaces is its weight W only; when it rests upon an inclined plane, as in Fig. 79, the normal pressure is $W \cos a$. If a body of weight W rests upon a horizontal plane and is subjected to a force P which makes an angle b with the vertical, as in Fig. 80, the normal pressure is $W + P \cos b$. In general the normal pressure N is the algebraic sum of the components of all the applied forces in a direction normal to the plane of contact of the two surfaces. The first law states that if this normal pressure be doubled, the friction F is also doubled; thus if a horizontal force of 1.6 pounds causes a brick to begin to move on a horizontal plane, a force of 3.2 pounds will be required when a second brick is placed upon the first one.

The second law states that, if the normal pressure remains the same, the friction is unaltered by varying the area of contact. Thus, a brick may rest upon a plane on its narrow or flat side or on its end, but the force required to cause sliding is the same in the three cases. This law applies to moderate pressures and is not valid for high ones.

The value of the coefficient of friction n may be determined by the method illustrated in Fig. 78, where the normal pressure is the weight W of the body, and the friction at the instant of sliding is measured by the weight P in the pan; the value of n is then P/W. For example, if a brick weighing 4.9 pounds begins to slide upon a horizontal plane under a horizontal force of 2.1 pounds, then the coefficient of friction is $n=2.1/4.9=0.43$.

Another method of determining the coefficient of friction is indicated in Fig. 79, where the plane AC is slowly raised around the hinge A until the body begins to slide. If a is the angle BAC, the normal pressure N is $W \cos a$ and the resisting friction F along the plane is equal to $W \sin a$. Accordingly the coefficient of friction is

$$n=F/N=W \sin a/W \cos a=\tan a$$

that is, the coefficient of friction is equal to the tangent of the angle BAC. The angle a at which the surface of contact must be inclined to the horizontal so that sliding just begins is often called the 'angle of friction'; this angle can be easily measured and its tangent is the coefficient of friction for the given surfaces. For example, if a brick begins to slide on a board when the board is inclined $16\frac{1}{4}$ degrees to the horizontal, the coefficient of friction is $n=\tan 16\frac{1}{4}^\circ=0.29$.

The following values of coefficients of friction give an idea of its range for different surfaces which are used in engineering practice:

for stone on stone, $n=$ from 0.50 to 0.65
for stone on timber, $n=$ from 0.30 to 0.50
for timber on timber, $n=$ from 0.25 to 0.40
for metal on metal, dry, $n=$ from 0.15 to 0.20
for metal on metal, oiled, $n=$ from 0.01 to 0.10

It is seen that these values are all less than unity, and that the effect of oil is to reduce the coefficient of friction and hence also the resisting force of friction. When it is desired to prevent sliding, the coefficient of friction should be high, and this will be the case if the surfaces are rough; when it is desired that sliding may easily occur, the coefficient of friction should be small, and this will be the case with smooth surfaces. Oil diminishes friction because it fills the depressions between the projecting points of the surfaces.

When the value of n is known, the horizontal force required to cause sliding to begin on a horizontal surface is easily computed, because this force is equal to the resisting friction F. For example, let a stone pier weighing 28 000 pounds rest upon a timber base, the coefficient of friction for these surfaces being 0.45; then the horizontal force required to cause sliding is $F=nN=0.45\times 28\,000=12\,600$ pounds.

When a body of weight W is pushed by an inclined force P which makes an angle b with the vertical, as in Fig. 80, sliding will begin if the values of P and b are sufficiently large. Here the normal pressure N is $W+P\cos b$ and the horizontal force F required to overcome the resisting friction is $P\sin b$. Hence the condition for sliding to begin is that $P \sin b$ shall equal $n(W + P \cos b)$. When the body is pulled by the force P, inclined at the angle b to the vertical, the normal pressure is $W - P \cos b$ and the horizontal force F is $P \sin b$, so that the condition for motion to begin is that $P \sin b$ shall equal $n(W - P \cos b)$. It is seen that the same inclined force P will start a heavier weight by pulling than by pushing.

PROBLEMS

181. A body weighing 43 pounds rests upon a table, and the coefficient of friction for the surfaces of contact is 0.28. Will it move under a horizontal force of 10 pounds?

182. A body weighing 47 pounds rests upon a table, and a force of 18 pounds acts horizontally upon it. If sliding just begins, compute the coefficient of friction.

183. A body begins to slide down a plane when it is inclined 24° 45′ to the horizontal. What is the coefficient of friction for the surfaces of contact?

184. If the coefficient of friction is 0.199, at what angle must the plane be inclined so that sliding begins?

185. A body weighing 47 pounds rests upon a table, and a force of 19 pounds making an angle of 3° 15′ with the horizontal pushes upon it, as in Fig. 80, and sliding begins. Compute the coefficient of friction.

186. For Fig. 80 let P be 100 pounds making an angle of 20° with the horizontal. If $n = 0.31$, what is the value of W so that sliding may begin?

187. For Fig. 81 let the data be the same and the value of W be required.

188. For Fig. 80 let W be 53.7 pounds, P be 46.0 pounds, and n be 0.42. What angle must P make with the vertical in order that the body may begin to slide?

189. The base of an inclined plane is 3 feet and its height

is 4 feet. What force, acting parallel to the plane, will just prevent a body weighing 28 pounds from sliding down, the coefficient of friction being 0.37?

190. The base of an inclined plane is 3 feet and its height is 4 feet. What force, acting parallel with the plane, will just cause a body weighing 28 pounds to begin sliding up the plane, the coefficient of friction being 0.37?

ART. 20. STABILITY AGAINST SLIDING

The resisting force of friction is an aid to the stability of all constructions which are required not to move. A house would be moved horizontally by the slightest wind and a dam would slide under the slightest water pressure if it were not for this resistance. The law $F=nN$ shows that stability against sliding may be increased by making the surfaces of contact rough so as to increase n, or by adding weight so as to increase N. To insure stability against sliding, it is necessary that nN shall always be greater than the greatest applied force which can act parallel to the surface of contact.

Another way of increasing the stability against sliding is to give the surface of contact such an inclination that the resultant R of all the applied forces makes a small angle with the normal to the plane. To explain this. let R in Fig. 82 be this resultant making the angle x with the normal. The normal pressure is $R \cos x$ and the force parallel with the surface of contact is $R \sin x$. Sliding will begin when $R \sin x = nR \cos x$, that is when $\tan x = n$. Let a be the angle whose tangent is n, this being called the angle of friction. Then the condition of sliding is $\tan x = \tan a$ or $x = a$. Hence sliding begins when the resultant of all the applied forces makes an angle with the normal to the plane equal to the angle of friction, and it does not occur when this angle is less than the angle of friction. The smaller this angle the greater will be the stability against sliding, and the highest degree of stability will be secured when the resultant R is normal to the plane of contact.

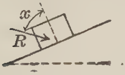

FIG. 82

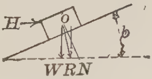

FIG. 83

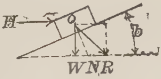

FIG. 84

As an application of this principle let a body of weight W rest upon a plane which is inclined to the horizontal at an angle b, this being greater than the angle of friction a. Let it be required to find the least horizontal force H which will prevent the body from sliding down this plane. Fig. 83 shows this case, where the resultant R makes an angle NOR with the normal ON equal to the angle of friction a; and since the angle NOW is equal to b, it follows that $H = W \tan(b - a)$. To find the least horizontal force which will cause the body to slide up the plane the resultant R is drawn on the other side of the normal ON, as in Fig. 84, and then $H = W \tan(b + a)$. The horizontal force which causes the resultant to coincide with the normal is $H = W \tan b$, and for this value there is no tendency whatever to slide. For instance, let b be 56° 15′, the weight 100 pounds, and the coefficient of friction n be 0.32, whence a is 17° 45′. Then, to prevent the body from sliding down, the least horizontal force is $H = W \tan 38° 30′ = 79.5$ pounds; to cause it to begin to slide up the plane, $H = W \tan 74° 0′$

$=348.7$ pounds; to keep it on the plane with the highest degree of stability, $H=W \tan 56° 15'=149.7$ pounds.

As another example, let Fig. 85 represent the cross-section of a dam on a horizontal base subject to a horizontal water pressure. Let P be 50 000 pounds, W be 120 000 pounds, and the coefficient of friction n be 0.62. Since 50 000 is less than $0.62 \times 120\,000$, the dam will not fail by sliding. Here also the value of $\tan x = WR/OW = 50\,000/120\,000 = 0.42$, so that x is less than a. To obtain a greater degree of security the base of the dam may be inclined as in Fig. 86 and an inclination can be computed so that the resultant OR shall be normal to AB, thus giving the highest possible security against sliding.

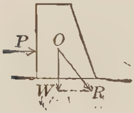

FIG. 85

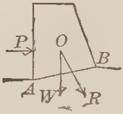

FIG. 86

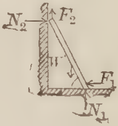

FIG. 87

As a final example, the interesting ladder problem will be discussed. Fig. 87 shows a ladder of length l and weight W resting with its foot upon the ground and its top against a wall, the angle of inclination to the horizontal being i. Let n_1 and n_2 be the coefficients of friction for the foot and top respectively, and let it be required to find the value of i so that sliding may begin. At the foot there act the upward pressure N_1 and the resisting friction F_1; at the top there act the horizontal pressure N_2 and the resisting friction F_2. From the first and second conditions of equilibrium in Art. 5,

$$N_2 - F_1 = 0 \qquad N_1 - W + F_2 = 0$$

and from the third condition, taking the weight of the ladder at its center of gravity, and an axis of moments at the top,

$$W \times \tfrac{1}{2}l \cos i - N_1 \times l \cos i + F_1 \times l \sin i = 0$$

Also, the law of friction in Art. 19 gives

$$F_1 = n_1 N_1 \qquad\qquad F_2 = n_2 N_2$$

These five equations contain the five unknown quantities N_1, N_2, F_1, F_2, and i. By solution the value of $\tan i$ is found to be

$$\tan i = (1 - n_1 n_2)/2n_1$$

For example, let the ground be rough so that $n_1 = 0.9$, and the wall be smoother so that $n_2 = 0.1$; then $\tan i = 0.51$, and $i = 27°$. The ladder will hence be stable if it is inclined more than 27 degrees to the horizontal, and its degree of stability will be the greater the larger the inclination.

PROBLEMS

191. The coefficient of friction between a brick and a board is 0.45. Will the brick slide if the board be inclined 37 degrees to the horizontal?

192. What horizontal force is required to push the brick of the last problem up the inclined plane, its weight being 5 pounds?

193. What force acting parallel with the plane will push the same brick up the same inclined plane?

194. A dam weighing 240 000 pounds rests on a horizontal base and is subject to a horizontal water pressure of 144 000 pounds. If the coefficient of friction is 0.61, what additional water pressure will cause failure?

195. In the dam of Fig. 86 the weight W is double the

horizontal pressure P, and the inclination of the base AB is 16 degrees. What angle does the resultant R make with the base?

196. Let the coefficient of friction for Fig. 83 be 0.52 and the values of H and W be equal. What must be the angle b so that W may begin to slide down the plane?

197. For Fig. 84 let $a = 24° 15'$ and $b = 65° 45'$. What is the least horizontal force which will cause W to begin to slide up the plane?

198. For the ladder in Fig. 87 show that N_1 is equal to W if either n_1 or n_2 is zero.

199. Show that the stability of a ladder is increased by making the lower part heavier than the upper part.

200. When a ladder is just about to slide under its own weight, its stability is increased if a man steps upon the lowest round, but if he ascends above the middle sliding will surely occur. How can this be demonstrated?

ART. 21. GRAVITY AND WORK

The earth exerts upon all bodies an attraction called the force of gravity, and it is this which causes bodies to press downward upon their supports, or to pull downward when they are suspended. The weight of a body is the measure of the force of gravity. When a body is lifted by a man, a constant downward resistance equal to its weight is experienced by him, and it is said in popular language that he does work in raising the body.

'Work' in mechanics is the product of a resistance by the distance through which it is overcome. Thus, in raising a weight W through the vertical distance h, the work Wh must be performed against gravity. It is found by experience that the amount of exertion required to lift a body from the position *A* to the higher level *BC* depends only upon the difference of level *h*. In Fig. 88 the body may be lifted vertically to *B* and the work *Wh* is performed; if it be moved along an inclined path to *C*, the same work *Wh* is performed.

The unit of work is called a foot-pound when the unit of resistance is the pound and the unit of distance is the foot. Thus, if a man lifts a body weighing 12 pounds through a vertical height of 6 feet, he performs $12 \times 6 = 72$ foot-pounds of work; if he raises it through a vertical distance of 1.5 feet, he performs 18 foot-pounds of work. In the metric system, where *W* is taken in kilograms and *h* in meters, the unit of work is called a kilogram-meter.

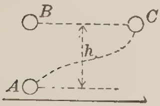

FIG. 88

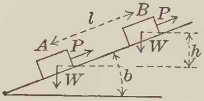

FIG. 89

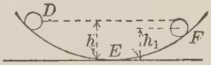

FIG. 90

When a body rests upon a level table and a horizontal force draws it along, there is no work done against gravity, for the body is not raised to a higher level. In this case the resistance is that of friction only (Art. 19).

Fig. 89 represents a body of weight *W* being drawn up an inclined plane which is so smooth that there is no frictional resistance. The length of the plane between the positions *A* and *B* is *l*, the vertical height of *B* above *A* is *h*, and the angle of inclination is *b*. The resistance of gravity in the line of motion is the component of *W* parallel to the plane or $W \sin b$, and $l \times W \sin b$ is the work done against gravity in moving the body along the plane from the position *A* to the position *B*. But since $l \sin b$ equals h, this becomes Wh. Hence the work required to be performed against gravity in drawing a body a certain distance up an inclined plane is the same as that required to lift it vertically through the same difference of level.

The work required to raise a body of weight W through the height h is independent of the time in which the operation is performed. The amount of work expended against gravity in lifting a body weighing 60 pounds through a height of 4 feet is 240 foot-pounds, whether the time be long or short, and the same work must be expended in raising [illegible] pounds through 8 feet, or 10 pounds through 24 feet, or [illegible] pounds through 120 feet.

Power is the measure of the capacity of a motor to perform work in a specified time and to continue working at that rate. The unit of power in common use is 33 000 foot-pounds per minute, and this is called a 'horse-power,' because a very strong horse can maintain this rate of working. A motor which has a capacity of one horse-power can perform 33 000 foot-pounds of work in each minute that it operates; that is, it can in each minute lift 33 000 pounds through a height of one foot, or one pound through a height of 33 000 feet, or 66 pounds through a height of 500 feet.

As an example, let it be required to find what horse-power engine is needed to lift every hour 2 400 000 pounds through a vertical height of 45 feet. The work to be done against gravity in one hour is $2\,400\,000 \times 45 = 108\,000\,000$ foot-pounds, and that in one minute is $108\,000\,000/60 = 1\,800\,000$ foot-pounds. The horse-power required is then $1\,800\,000/33\,000 = 54\frac{1}{2}$; this power overcomes the force of gravity only, and additional power is needed to overcome the frictional resistances.

When a body of weight W falls through the vertical height h, gravity is said to store up work in it and the amount of this work is Wh. If none of this work is lost in frictional resistances, it can be utilized to lift the body again to the level from which it started. This fact has been ascertained by experiments conducted as shown in Fig. 90, where the body at the position D is guided in its descent by the smooth curve DEF. On arriving at the lowest point E, gravity has stored in the body the work Wh and the body is moving with a certain velocity; it then ascends the curve EF until it arrives at the position F. Such experiments show that the height h_1, to which it rises above E, is always less than the height h of fall, and that the smoother the curve the nearer does h_1 approach h. It is hence concluded that if all frictional resistances are absent, the work stored in the body at E is sufficient to raise the body through a height equal to the height of fall. This experiment should be tried by the student; a ball hung at the end of a string may be used instead of the apparatus above described.

PROBLEMS

201. How many foot-pounds of work are required to lift a body weighing 108 pounds through a height of $11\frac{3}{4}$ inches? What horse-power is required to lift this weight 150 times in one minute?

202. A gallon of water weighs $8\frac{1}{3}$ pounds. What horse-power is required to lift one million gallons in eight hours through a vertical height of 125 feet?

203. The base of an inclined plane is 40 feet and its height

is 30 feet. If the plane be perfectly smooth, what work must be performed to raise a body weighing 100 pounds from the foot to the top?

204. How many gallons of water can be lifted in one hour through a height of 76 feet by a pumping-engine of 83 horse-powers, if no work is lost in frictional resistances?

205. A cubic foot of water weighs $62\frac{1}{2}$ pounds. What horse-power can be generated by 800 cubic feet of water falling in every second from a height of 13.6 feet, if no work is lost in frictional resistances?

206. If a motor lifts 550 pounds of matter in every second through a height of one foot, what is its horse-power?

207. In Fig. 90 let D be 5 feet above E and let the curve be so rough that 20 percent of the work of the fall is lost in friction. To what height above E will the body ascend?

208. In the metric system one horse-power is taken as 4500 kilogram-meters of work per minute. How many horse-powers are required to lift 500 cubic meters of water in one hour through a height of 60 meters?

209. What forces tend to stop the swinging of the pendulum of a clock, and how is it kept in motion?

210. A man weighing 150 pounds runs up the stairs of a four-storied building in 78 seconds. If the vertical height of ascent is 48 feet, what work does he perform and what is his horse-power?

ART. 22. WORK AGAINST FRICTION

When a body begins to slide upon a plane, the resisting friction F is given by nN, where n is the coefficient of friction and N is the normal pressure (Art. 19). The same law applies when the body is sliding, but the value of the coefficient of friction n is a little less than at the moment of starting. This fact has been ascertained by experiments, and moreover it is a familiar fact of observation that the force required to cause a sled to begin to move is greater than that needed to keep it moving at a moderate velocity.

If the value of n for sliding friction is given, the resistance F, in pounds, is computed by multiplying n by the normal pressure N, and the work expended in moving through the distance l is then Fl. If the motion be uniform through a distance of l feet and occupy t minutes, the work expended per minute is Fl/t, and this divided by 33 000 gives the horse-power of the animal or motor which maintains the motion. For, by the definitions of Art. 21, work is resistance multiplied by the distance through which that resistance is overcome, and one horse-power is 33 000 foot-pounds of work per minute.

For example, let a body of 14 200 pounds weight be kept sliding at the rate of 25 feet per minute on a horizontal plane, the coefficient of sliding friction being 0.18. The resisting friction is $0.18 \times 14\,200 = 2556$ pounds, and the horse-power expended is $2556 \times 25/33\,000 = 1.94$ horse-powers.

When a body is pulled up an inclined plane making the angle b with the horizontal by a force P parallel with the plane, as in Fig. 91, the resistance is due partly to friction and partly to gravity. The weight being W, the normal pressure is $W \cos b$ and the frictional resistance is $nW \cos b$; the resistance due to gravity is $W \sin b$, this being the component of W parallel to the plane. Then the work K performed in passing from A to C is

$$K = (nW \cos b + W \sin b) \times AC$$

Since $AC \cos b$ is the base AB, and $AC \sin b$ is the height BC, this equation becomes

$$K = nW \times AB + W \times BC$$

in which the term $nW \times AB$ is the work required to overcome friction through the horizontal distance AB, and $W \times BC$ is the work required to overcome gravity through the vertical distance BC. Hence the work required to pull a body up an inclined plane is the same as that required to draw it along the horizontal projection of the plane plus that required to lift it through the vertical projection of the plane.

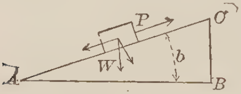

FIG. 91

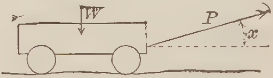

FIG. 92

When a vehicle is in motion it meets with resistances which tend to stop it, such as friction at the axles, friction of the air, and oscillations due to irregularities of the ground. The sum of all these resistances is called 'traction,' and the laws of traction are closely the same as those of sliding friction, except that it is not independent of the area of contact of the wheel upon the ground. For the same vehicle, however, the law $F = mN$ is true, where F is the resisting traction, N is the normal pressure, and m is an abstract number called the 'coefficient of traction.' The value of m is about 0.10 for a common earth road, about 0.08 for a gravel road, about 0.04 for a macadam road, about 0.015 for an asphalt pavement, and about 0.003 for slow speed on a railroad track. By using these values of m all the above principles relating to the work of sliding friction may be applied to the traction of vehicles.

For example, let it be required to find what horse-power is needed to draw a wagon weighing 3600 pounds on a level macadam road at a speed of $4\frac{1}{2}$ miles per hour. The traction resistance is $0.04 \times 3600 = 144$ pounds, and the distance passed through in one minute is $4\frac{1}{2} \times 5280/60 = 396$ feet. The work to be expended in one minute is then $144 \times 396 = 57\,024$ foot-pounds, and the power needed is $57\,024/33\,000 = 1.73$ horse-powers. If the same road have a vertical rise of 3 feet for every 100 feet of horizontal distance, the weight of 3600 pounds must be lifted 11.88 feet in each minute and there must be performed in each minute the work of $3600 \times 11.88 = 42\,768$ foot-pounds, which requires $42\,768/33\,000 = 1.30$ horse-powers additional, while that required to overcome friction remains as before at 1.73 horse-powers.

When a body is drawn down an inclined plane the effect of gravity is to diminish the traction resistance. Thus, if the wagon of the last paragraph is pulled on a down grade of 3 feet per 100 feet, the horse-power required to maintain the motion of $4\frac{1}{2}$ miles per hour is $1.73 - 1.30 = 0.43$ horse-powers. For a slightly steeper grade the component of the force of gravity parallel with the plane becomes greater than the traction resistance, and the body will move down the plane without an applied pull unless the brake be applied.

It is well known that the shafts of a wagon should be inclined to the horizontal at a small angle x, as in Fig. 92, in order that the horse may pull with the least effort. On a level road the normal pressure for this case is $W - P \sin x$ and the traction resistance is $m(W - P\sin x)$.

Under the condition of starting, or of uniform motion, this resistance equals the horizontal pull $P \cos x$. The value of P is now found to be $mW/(\cos x + m \sin x)$, and this is the force exerted by the horse through the shafts. It may be shown by a branch of mathematics called the differential calculus that P will have the smallest value when $\tan x$ equals m. Hence the rougher the road the greater should be the angle x. For a common earth road, $\tan x$ should be about 0.1, that is, the inclinaton of the shafts should be between 5 and 6 degrees in order that the pull may be a minimum.

PROBLEMS

211. Compute the horse-power required to keep a body sliding on a horizontal plane at the rate of 120 feet per minute, its weight being 9500 pounds and the coefficient of friction being 0.17.

212. Compute the horse-power required for the above data when the body is sliding up a plane which has an inclination to the horizontal of one foot vertical in 10 feet of horizontal distance.

213. Compute the horse-power for the above case when the body is sliding down the inclined plane.

214. Find the weight that can be drawn on a common earth road by a motor of 5 horse-powers at a speed of 6 miles per hour.

215. Find the weight that can be drawn on a railroad track by a motor of 50 horse-powers at a speed of 6 miles per hour.

216. How many miles per hour can be maintained with a wagon weighing 2400 pounds on a level macadam road, when it is drawn by two small horses capable of performing $1\frac{1}{4}$ horse-powers?

217. Compute the values of $\cos x + 0.10 \sin x$ for the following values of $\tan x$: 0.08, 0.09, 0.10, 0.11, 0.12.

218. Compute the value of P for Fig. 92, when $W = 2800$ pounds and $m = 0.06$, the shaft having a rise of 2 vertical inches in 50 horizontal inches; also when it rises 3 inches in 50 inches; also when it rises 4 inches in 50 inches.

219. A horizontal force of 500 pounds, making an angle of 30 degrees with a level railroad track, pulls a car at uniform speed through a distance of 125 feet in one minute. What work is done per minute?

220. At high speeds the coefficient of traction on a railroad track is greater than the value above given, it being about 0.012 for 50 miles per hour. What is the horse-power of a locomotive which will pull six coaches, each weighing 60 000 pounds, at this speed up a grade of six inches rise in 100 horizontal feet?

CHAPTER V

SIMPLE MACHINES

ART. 23. THE LEVER

A lever is a machine by which a force applied at one point may be transferred so as to produce pressure or pull at another point. In its simplest form it consists of a very light straight horizontal bar to one end of which the vertical downward force P is applied in order to support the weight W at the other end, as in Fig. 93. The lever rests upon a fixed point called the 'fulcrum,' and the moment of P must be equal to the moment of W with respect to an axis at this fulcrum when equilibrium is maintained (Art. 8). Accordingly, if b and c are the lengths of the arms of the lever on the left and right of the fulcrum,

$$Wc - Pb = 0 \qquad \text{or} \qquad W = P \, . \, b/c$$

which shows that W is greater than P when b is greater than c. Thus, if P is 20 pounds, b is 48 inches, and c is 3 inches, the weight W which can be supported is 320 pounds.

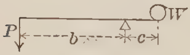

FIG. 93

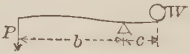

FIG. 94

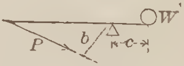

FIG. 95

The lever may be curved as shown in Fig. 94, but the same equation holds if b and c are the distances from the fulcrum to the lines of direction of P and W; that is if b and c are lever arms of these forces (Art. 7). When the force P is inclined to the vertical, as in Fig. 95, the same equation of moments is true if b is the length of the lever arm from the fulcrum normal to the direction of P. In all cases, then, the forces at the ends of the lever are inversely as their lever arms.

Another kind of lever is that in which the supported weight W is between the ends, the fulcrum being at one end and the applied force P acting upward at the other. In Figs. 96 and 97 the force P acts vertically, the lever being straight in the first case and bent in the second. In Fig. 98 the force P is inclined to the vertical. In each of these cases let l be the lever arm of P, and m be that of W with respect to the fulcrum; then

$$Pl - Wm = 0 \qquad \text{or} \qquad W = P.l/m$$

so that W is greater than P when l is greater than m. If P be 20 pounds, l be 51 inches, and m be 3 inches, then W is 340 pounds.

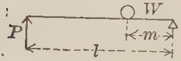

FIG. 96

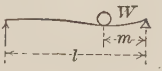

FIG. 97

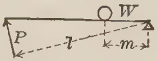

FIG. 93

A lever may be also used as a machine for lifting a weight through a small vertical height. For example, let the force P in Fig. 99 pull down the left end of the lever through the vertical height h and thus overcome the resistance of the weight W as it rises through the vertical height k. The work Ph is thus applied to the machine, and the work Wk is utilized in overcoming the force of gravity. If no work is lost in friction or in bending the bar, then $Wk = Ph$. Let b and c be the lever arms of P and W when the bar is horizontal; then from similar triangles $h/k = b/c$, and hence the equation becomes $Wc = Pb$, which is the law of the lever as deduced above. But it is found that work is always lost in friction and other resistances, so that the delivered work Wk is less than the applied work Ph, and hence Ph must be greater than Wk, or P must be greater than $W.c/b$.

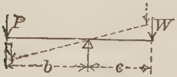

FIG. 99

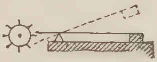

FIG. 100

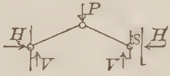

FIG. 101

In the above discussions the weight of the lever itself has not been considered, but this will now be taken into account. When the bar is of uniform size, let w be its weight per linear unit and l its length $b+c$, so that its total weight for Fig. 93 is wl. The center of gravity of this weight is at the distance $b - \frac{1}{2}l$ from the fulcrum, so that the equation of moments gives

$$Pb + wl(b - \tfrac{1}{2}l) = Wc, \qquad \text{or} \qquad W = P\frac{b}{c} + \frac{wl(b - \frac{1}{2}l)}{c}$$

Similarly for Fig. 96 it is easily found that the value of W required for equilibrium is $(Pl - \frac{1}{2}wl^2)/m$. For Fig. 93 the weight of the bar has no influence, if its center of gravity lies vertically above the fulcrum, but for Fig. 96 the weight of the bar always resists the applied force P. When P and W are large compared with wl, it may not be necessary to take account of the weight of the lever itself, and this is the assumption made in the previous paragraphs. A beam resting on supports at its ends is in all respects like Fig. 96, the force P being the reaction of the left support.

PROBLEMS

221. The light bar shown in Fig. 93 is 5 feet long and the weight W is 100 pounds. Compute the values of P when the arm c has the lengths 1, 2, 3, 4 feet.

222. Compute the values of P for the above data, taking the weight of the bar as 10 pounds.

223. The light bar shown in Fig. 96 is 5 feet long and the applied force P is 50 pounds. Compute the values of W when the arm m has the lengths 1, 2, 3, 4 feet.

224. What vertical pressure comes on the fulcrum of Fig. 94 when P is 20 pounds, b is 4 feet, and c is 1.5 feet?

225. A bar weighs one pound and is so shaped that its center of gravity lies one inch to the left of the fulcrum (Fig. 93). A weight P may be moved along the arm b so as to balance a load W placed at the right end. If P is 0.752 pounds and c is 3.375 inches, where must P be placed when W has the values 0, 5, and 10 pounds?

226. In Fig. 95 let P be a force of 30 pounds inclined 17° 15′ to the vertical. If the horizontal length of the light bar is 85 inches, what weight W can be supported at a distance of 12 inches from the fulcrum?

227. For Fig. 96 let l be 62 and m be $10\frac{1}{2}$ inches. How high will W be lifted when the end at P is raised $6\frac{1}{4}$ inches?

228. Three men carry a stick of timber, one man taking hold at one end, and the others at a common point. Where is this point so that each man may bear one-third of the weight?

229. A beam 12 feet long weighs 40 pounds and carries a load of 37 pounds at the right end. One support is at the left end and the other may be moved. Where shall it be put so that the pressure on the left support may be one pound?

230. Fig. 100 represents a pounding-machine, the hammer having a weight of 192 pounds and its center of gravity being 4 feet from the fulcrum. The short arm of the lever is 3 inches long and it is depressed one inch, four times per second, by

the revolution of the wheel A. What is the greatest horsepower that can be delivered by the hammer?

ART. 24. SYSTEMS OF LEVERS

A system of levers formed by connecting three simple levers, as in Fig. 102, may be used when it is desired to balance a heavy weight by a light one. There are three fixed fulcrums A, A_1, A_2; the three lever arms b, b_1, b_2 are long, while the lever arms c, c_1, c_2 are made as short as practicable. The weight W causes an upward pressure R_1 to be exerted at C, and this in turn causes the downward pressure R at B, and this is balanced by the force P. From the last article, the value of R_1 is $W \cdot c_2/b_2$, that of R is $R_1 \cdot c_1/b_1$ or $W \cdot c_1c_2/b_1b_2$, and that of P is $R \cdot c/b$ or $W \cdot cc_1c_2/bb_1b_2$. Thus if c is one-tenth of b, c_1 is one-tenth of b_1, and c_2 is one-tenth of b_2, the force P will be one-thousandth of W.

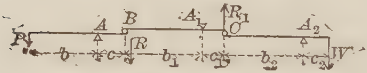

FIG. 102

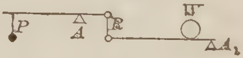

FIG. 103

It is on this principle that scales for weighing heavy loads are constructed. Fig. 103 shows a simple form where the lower lever is 12 feet long and its fulcrum is at the end, W being 1000 pounds at a distance of $1\frac{1}{2}$ feet from this fulcrum, while the short arm of the upper lever is 6 inches long and the load P is 20 pounds. To ascertain where P should be put in order to secure equilibrium, the force R in the vertical link is first found to be 125 pounds; then if b is the distance of P from A, the equation of moments is $20b = 125 \times 6$, whence $b = 37.5$

inches. From such computations the scale arm may be graduated, but to insure accuracy it will be necessary to take into account the weight of each lever. This form of scale is, however, not a convenient one in practice, for the load W must always be applied at the same point on the lower lever.

Fig. 104 shows levers of a platform scale so arranged that the reading of the scale arm AL is the same wherever the load W may be placed on the platform. This result is secured by making the horizontal distances CD and BE equal, as also CF and BG. In this machine there are three fixed fulcrums A, B, C, and the platform rests upon the lever CF at D and upon the lever BH at G, while the vertical link FG connects these two levers. Let CD and BE be represented by p, and CF and BG by q, BH by h, AK by k, and AL by l. Let W be placed on the platform so that the horizontal distances from it to D and E are d and e. The part of W borne at D is $W.e/(d+e)$, the part of this carried to F is $W.e/(d+e).p/q$, and the part of this carried to H is $W.e/(d+e).p/q.q/h$. The part of W borne at E is $W.d/(d+e)$, and the part of this carried to H is $W.d/(d+e).p/h$. The sum of the parts carried to H is then $W.p/h$, which is independent of d and e and is the same value as if the entire weight had been applied at E. Then $P.l/k = W.p/h$ gives the law of this system of levers. For example, let P be 2 pounds, AL be 18 inches, AK be one inch, BE be one inch, and BH be 30 inches; then the load W will be 1080 pounds.

Fig. 105 shows the system of levers called the Roberval balance, which is often used for pan balances, because it is immaterial in what part of the pan the weights are

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placed. AB and CD are horizontal bars pivoted at the middle points E and F, while CG and DH are vertical bars rigidly attached to the two scale-pans. There are joints at A, B, C, D, so that the rectangle $ABCD$ can easily change into a parallelogram when one end rises and the other falls. If equal weights be placed anywhere in the scale-pans, they are transferred to AC and BD and the bars remain horizontal. For example, the two weights P shown in the figure will exactly balance notwithstanding that their distances from E are different; if the lower bar CD is removed, the end A will immediately fall. It will be well for the student to make one of these balances with seven sticks and a few nails.

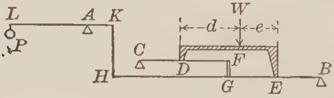

FIG. 104

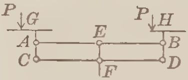

FIG. 105

PROBLEMS

231. Find for Fig. 102 the vertical pressures borne by each of the fulcrums and show that their sum is $P+W$.

232. Find the vertical pressures on each of the fulcrums in Fig. 103, using the data given in the second paragraph.

233. If a lever like Fig. 93 is used for weighing by moving P along the arm b, show by Art. 16 that the center of gravity of the arm and loads should lie below the fulcrum.

234. In Fig. 102 let $b_2/c_2=b_1/c_1=10$, and c be 2 inches. What is the value of b if $P=3$ pounds and $W=2400$ pounds?

235. For Fig. 104 let $AK=BE=CD=1\frac{1}{4}$ inches, $CF=BG=18\frac{1}{8}$ inches, and $BH=30\frac{3}{4}$ inches. Find the value of AL when $P=5$ and $W=1000$ pounds.

236. Using the same lengths of arms find the value of AL when $P=5$ and $W=2000$ pounds.

237. Consult an advanced work on mechanics and explain the theory of the Roberval balance.

238. A man standing on the platform of a weighing-machine presses downward upon the platform with his cane. Will this cause his weight to appear greater?

239. A straight lever 72 inches long has weights of 3 and 31 pounds hung from its ends, and it balances upon a fulcrum distant 13 inches from one end. Compute the weight of the lever.

240. A straight beam inclined at an angle of 89 degrees to the vertical is held in equilibrium by a vertical force at its foot and by horizontal forces of 800 pounds at both foot and top. Compute the weight of the beam.

ART. 25. THE INCLINED PLANE

The inclined plane is often used in lifting a weight from a lower level AB to a higher level CD, as in Fig. 106, in order to diminish the pull P. If the weight is lifted vertically from A, this pull will be equal to W, the weight of the body, but by the help of the inclined plane it can be made much less. Let b be the angle which the plane AC makes with the horizontal, and x the angle which P makes with the plane AC. If there is no friction on the plane, the value of P is found by equating to zero the sum of the components of the forces parallel to the plane, which gives

$$P \cos x - W \sin b = 0 \qquad \text{or} \qquad P = W . \sin b / \cos x$$

and this shows that both b and x should be small angles in order to render the pull P small.

Friction, however, is always a resisting force when a body slides up a plane. Let n be the coefficient of friction and N the normal pressure on the plane, then the resisting friction is nN (Art. 19). The normal pressure for Fig. 106 is $W \cos b - P \sin x$, and the resisting friction is $n(W \cos b - P \sin x)$, which must equal the resultant moving force $P \cos x - W \sin b$. From this equality, there is found

$$P = W \frac{\sin b + n \cos b}{\cos x + n \sin x}$$

When W and n are given, P may be computed for assigned values of b and x. For example, let W be 1000 pounds and n be 0.3; then for $b = 30°$ and $x = 16° \, 45'$, the value of P is 728 pounds. The value assumed here for x is that which gives the largest value to the denominator of the fraction (Art. 22).

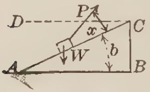

FIG. 106

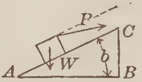

FIG. 107

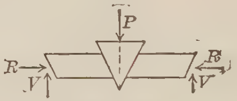

FIG. 108

When the force P is inclined downward at an angle x with a line parallel to the plane, as in Fig. 107, the normal pressure is increased and the pull becomes much greater. It is easily shown that the above formula applies to this case if the sign of the last term of the denominator is negative. For the above numerical data the value of P for this case is 872 pounds.

The inclined plane, when used as aid in lifting weights, has no mechanical advantage except in diminishing the pull necessary to move the body. The work expended is always greater than when the body is lifted vertically from the lower to the higher level (Art. 21), for the resistance of both gravity and friction must be overcome.

Two inclined planes placed together so as to form an isosceles triangle, as in Fig. 108, constitute an apparatus called a 'wedge.' It is used to generate lateral pressures by applying a force P upon the base of the wedge. Let V and R be the resisting forces parallel and normal to the direction of P, and let b be one-half of the angle at the vertex of the wedge. Then the relation between R and V is exactly the same as that between the weight W and the horizontal force H on a plane inclined to the horizontal at the angle b. Hence, if there is no friction on the sides of the wedge,

$$V = \tfrac{1}{2}P \qquad \text{and} \qquad R = \tfrac{1}{2}P \cot b$$

are the equations of equilibrium. Accordingly, if b is a small angle, the lateral pressure R will be large compared with P: for example, let b be one degree and P be 10 pounds, then R will be 286 pounds.

The frictional resistances along the sides of the wedge diminish the above value of R, this case being identical with that in Fig. 84, where W and H correspond to the R and V of Fig. 108. Hence, for the wedge under actual conditions, the values of the resisting forces are

$$V = \tfrac{1}{2}P \qquad \text{and} \qquad R = \tfrac{1}{2}P \cot(b + a)$$

where a is the angle of friction, namely, the angle whose tangent is equal to the coefficient of friction n. Thus, if n is 0.114, the angle a is 6° 30′; then if b is one degree and P is 10 pounds, the value of R is 38 pounds.

PROBLEMS

241. On a plane of 16° 45′ inclination a body weighing 100 pounds is held in equilibrium by a horizontal force. Find its value, if there is no friction.

242. On the same plane let the coefficient of friction be 0.24. Find the least horizontal force which prevents the body from sliding down.

243. For the same plane find the least horizontal force which will cause the body to slide up.

244. Show that the resultant of V and R for the wedge is $\frac{1}{2}P/\sin b$ when there is no friction.

245. Find the resultant of V and R for the wedge when the angle of friction for the sliding surfaces is a.

246. Using the numerical data in the second paragraph, find the number of foot-pounds of work required to raise the body up the inclined plane, if its length is 80 feet.

247. What horse-power is expended in lifting 4000 pounds per minute up a plane of 27° 15′ inclination and 275 feet length if there is no friction?

248. What horse-power is expended under the same conditions if the coefficient of friction is 0.33?

249. For the wedge let P be 100 pounds, R be 400 pounds, and the coefficient of friction be 0.12. Compute the value of b so that motion occurs.

250. For Fig. 107 let the horizontal force P be 8000 pounds and the load W be 18 000 pounds, while the wedge ABC weighs 38 000 pounds. The base AB is 32 feet and the height BC is 24 feet. Will sliding occur on the base, if the coefficient of friction is 0.37?

250*a*. Consult Ball's Experimental Mechanics (London, 1871), and ascertain how accurately the coefficients of friction may be found by the help of the inclined plane.

ART. 26. THE SCREW

Fig. 109 shows a common square-threaded screw which can revolve in the fixed nut N. The force P is here applied to a bar inserted in the head, this force acting in a plane normal to the axis of the screw. When the screw is turned by the force P, acting with lever arm l with respect to the axis, the screw will advance in the fixed nut toward the right and overcome a resisting force R at the right end; under these circumstances it is required to find the relation between P and R.

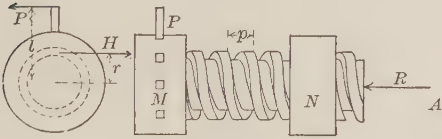

FIG. 109

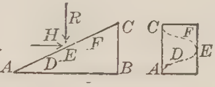

FIG. 110

When a right-angled triangle has a base equal to the circumference of a cylinder and is wrapped around that cylinder, its hypothenuse forms a curve on the cylindrical surface which is called a 'helix.' Thus in Fig. 110 let r be the radius of a cylinder and $2\pi r$ its circumference; then, if AB is equal to $2\pi r$, the line AC will form the helix $ADEFC$ on the cylindrical surface and the inclination of the helix will be at all points the same as that of the inclined plane AC. All the lines of the screw shown in Fig. 109 are helixes, those on the outer cylinder having a less inclination than those on the inner cylinder. The distance AC in the right-hand diagram of Fig. 110 is called the 'pitch' of the helix, this being the altitude of the right-angled triangle whose base is the circumference of the cylinder. In Fig. 109 all the helixes have the same pitch p, this being the distance which the screw will advance in the fixed nut when one revolution is made by means of the applied force P.

Let r be the mean radius of the helical surface; then its mean inclination is the angle whose tangent is $p/2\pi r$, for in Fig. 110, BC represents p and AB represents $2\pi r$. Let this angle be called b. The horizontal force H required to move the resistance R up the inclined plane AB is $R \tan b$ if there is no friction, or $R \tan(b+a)$ if a is the angle of friction (Art. 20). Now this force H acting with the lever arm r must balance the applied force P acting with the lever arm l, or $Hr=Pl$. Consequently

$$P=\frac{r}{l} R \tan b \qquad\qquad P=\frac{r}{l} R \tan(b+a)$$

are the conditions for equilibrium of the screw, the first when there is no friction, and the second when friction is taken into account.

Another method for deducing the equation for the screw without friction is as follows. Let the applied force P cause the screw to advance the distance p in one revolution against the resistance R. The applied work is $P \, . \, 2\pi l$ and the work performed is $R \, . \, p$. Accordingly $P \, . \, 2\pi l=Rp$, and this is the same as above, since p equals $2\pi r \, . \tan b$. This may be written

$$R=P \, . \, 2\pi l/p \quad \text{or} \quad P=R \, . \, p/2\pi l$$

which shows that R may be made very great by making the pitch p very small. Thus, if P is 10 pounds acting at a distance of 3 feet from the axis, and if the pitch of the screw is $\frac{1}{8}$ inch, then the resistance R is about 18 000 pounds. But the influence of friction will much diminish this value, as the following exercises show.

The above method of applying the resistance to the end of the screw while the nut is fixed is that used in the common screw press for copying letters, and in many machines for testing materials. Another method, less often used, is to have the nut movable and apply the resistance to it while the screw is prevented from advancing; the nut then moves longitudinally as the screw turns, and the same equations as above give the relation between P and R.

PROBLEMS

251. If the mean radius r is $1\frac{1}{4}$ inches and the pitch of the screw is $\frac{1}{8}$ inch, find the angle b.

252. What force P acting at a distance of 6 inches from the axis is necessary, for the last problem, in order to overcome a resistance of 6000 pounds, if there is no friction?

253. What force P is necessary for this case when the coefficient of friction for the helical surfaces is 0.01?

254. When the force R is applied to the end of the screw show that backward motion will occur if Pl is less than $Rr\tan(b-a)$.

255. Using the data in the preceding problems, compute the least force P which will prevent backward motion when a force R of 6000 pounds is applied at the end.

256. Under what circumstances is it advantageous to lubricate the threads of a screw, and under what circumstances might it be disadvantageous?

257. Consult a more advanced work on mechanics and ascertain what is meant by a differential screw.

258. If the screw has 24 threads per foot of length and its mean radius is 3 inches, compute the angle b.

259. For this screw compute the distance from the axis at which a force of 25 pounds must act in order to overcome a resistance R of 12 000 pounds, if there is no friction.

260. Solve the last problem if the coefficient of friction for the helical surfaces is 0.08.

ART. 27. THE PULLEY

A wheel which is turned by a cord or belt passing around part of its circumference is called a 'pulley.' It may revolve upon or be fastened to its axis, but in the latter case the axis can turn in its supporting bearings. A single pulley may be used in lifting a weight by an applied force as in Fig. 111, the weight W being at one end of the cord and the force P applied at the other end. The cord being supposed to be perfectly flexible, the tension in it will be uniform throughout, and hence $W = P$, if there is no friction on the axis of the pulley. The advantage of the single fixed pulley is in permitting the force P to be exerted in a convenient direction.

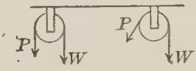

FIG. 111

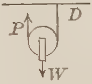

FIG. 112

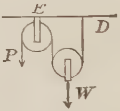

FIG. 113

A movable pulley with the weight attached to its axis is shown in Fig. 112, one end of the cord being held fast at D and the force P acting vertically upward. Here the tension in the cord is P on both sides of the pulley, and hence $W = 2P$, or an applied force P can support a weight equal to $2P$. The direction of the pull is, however, unfavorable, since it is harder for a man to pull upward than to pull downward, and Fig. 113 shows how this disadvantage may be overcome.

The above relations between P and W may be verified by the principle of moments, taking the axis of moments at the center of the axis of the pulley. Let r be the radius of the pulley, then for Fig. 111 equilibrium occurs when $Wr = Pr$, that is when $W = P$. For Fig. 112, taking the axis at D, the equation is $P \times 2r = W \times r$, whence $W = 2P$. For Fig. 113 it must be noted that an upward force $2P$ is exerted at E on the axis of the fixed pulley, so that the equation of moments with respect to D is $2P \times 3r - P \times 4r - W \times r = 0$, which gives $W = 2P$.

For the fixed pulley in Fig. 111 the distance which W is lifted is the same as that through which P is exerted, but for the movable pulley the distance is only one-half as great. This may be shown as follows from the idea of work (Art. 21). Let d be the distance through which P is exerted in overcoming the resistance W, and let x be the height through which W is lifted. Then, if no work is lost in friction or in overcoming the rigidity of the cord, $Pd = Wx$; for Fig. 111, W equals P, whence $x = d$; for Fig. 112, however, W equals $2P$, and hence $x = \frac{1}{2}d$.

Various systems of pulleys are shown in the following figures, the most common being that of Fig. 114, where the weight is attached to a movable block. If there be two pulleys in the fixed block and two in the movable one, and the tension in the cord is uniform throughout, the weight W equals $4P$, since it is supported by four cords, but in its rise will be only one-fourth of the distance through which P is exerted. If there are three pulleys in each block, W equals $6P$.

In Fig. 115 there are two movable pulleys; the tension in the cord around the lower one is $\frac{1}{2}W$, and that in the other cord is $\frac{1}{4}W$; accordingly $W = 4P$ for this case. For the arrangement in Fig. 116, the tension in the cord around the lower pulley is P, while that in the other cord is $2P$; hence $W = 3P$.

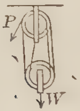

FIG. 114

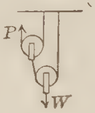

FIG. 115

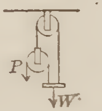

FIG. 116

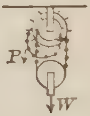

FIG. 117

Fig. 117 shows the device called a differential pulley, by which a small force P is enabled to lift a heavy weight W. The two pulleys in the upper fixed block are in one piece and turn about a common axis, each having pins on its circumference to prevent the chain or cord from slipping. Let R be the radius of the larger pulley, and r that of the smaller one. Then, since the tension in each branch of the chain around the lower movable pulleys is $\frac{1}{2}W$, the equation of moments with respect to the axis of the fixed pulley is $\frac{1}{2}W \times R - \frac{1}{2}W \times r - PR = 0$, from which is found $W = 2PR/(R-r)$. This shows that W can be made very large by making $R-r$ very small, and in fact the weight which can be lifted is only limited by the strength of the chains and pins.

In all the above cases the pulleys and cords have been supposed to be without weight, the cords to be perfectly flexible, and all frictional resistances to be absent. These conditions, however, never exist, and hence the weight W which can be supported or lifted by a given force P is always less than that given by the above equations.

PROBLEMS

261. In Fig. 113 each pulley weighs 10 pounds and the applied force P is 12 pounds. What weight can be lifted if there is no friction and the cord is perfectly flexible?

262. In a system like that of Fig. 114 there are four pulleys in each block. What is the relation between P and W?

263. Find the relation between P and W for a system like that of Fig. 115 when there are four movable pulleys.

264. Find the relation between P and W for a system like that of Fig. 116 when there are five pulleys.

265. In a differential pulley let the two radii be 6 and $5\frac{3}{4}$ inches. What force P is required to lift a weight of 480 pounds?

266. In a system like Fig. 114 there are three pulleys in each block, and a weight P of 6 pounds balances a weight W of 30 pounds. Find the weight of the lower block.

267. Draw a sketch like Fig. 114, with three pulleys in the upper block and two in the lower one, the end of the cord being attached to the lower block. Find the relation between W and P.

268. Let the lower block of the last problem weigh 35 pounds, and a man weighing 150 pounds hang upon it with one hand. What tension must he exert upon the free end of the string with the other hand in order to maintain equilibrium?

269 and 270 may be exercises taken from other books.

ART. 28. EFFECT AND EFFICIENCY

The 'effect' of a force acting upon a body is the pressure or the velocity which it produces (Art. 2). The effect of a force applied at any point of a machine is the pressure produced at another designated point or the distance through which that point is moved against a resistance. Thus, in a simple lever like Fig. 99 the object of the applied force P may be to cause a pressure W at the other end; in this case no motion occurs and the pressure W equals $P.b/c$. If, however, it is required to move the right end through a vertical distance k, the left end must be depressed the vertical distance h, and k equals $h.c/b$; in this case the applied work Ph equals the delivered work Wk, if no work is lost in overcoming other resistances. On account of these resistances the useful work Wk is smaller than the applied work. Now, for any given value of Wk, let the resistance W be increased, then the distance k will be diminished; or if the distance k be increased, then will the resistance W be diminished. One effect of a machine cannot therefore be generally increased without altering the other; it is in fact a well-known axiom in shops that 'what is gained in force is lost in speed.'

The 'efficiency' of a machine is the ratio of the useful work performed by it to the total work expended. Thus, for the simple lever of Fig. 100, let the useful work performed at the long end be 522 foot-pounds per minute, and the work applied at the other end be 600 foot-pounds per minute; then the efficiency of the machine is $522/600 = 0.87$, that is, the machine has utilized 87 percent of the work delivered to it.

The efficiency of raising a weight by the help of the inclined plane is readily determined from the discussion of Art. 22. Let b be the angle of inclination of the plane and l its length, n the coefficient of friction or traction, and W the weight which is raised through the vertical height $l \sin b$. Then the useful work performed is $Wl \sin b$

and the total work expended is $Wl\sin b + nWl\cos b$. Hence the efficiency of the inclined plane is given by

$$e = \frac{Wl\sin b}{Wl\sin b + nWl\cos b} = \frac{1}{1+n\cot b}$$

For example, let n be 0.25 and b be 18 degrees; then $e=1/1.77=0.565$, that is, 56½ percent of the total work is utilized in useful work. This formula applies also to the screw; if the pitch is small, b will be a small angle and cot b will be large, and hence n must be made small by lubricating the threads in order to render the efficiency large.

The efficiency of systems of levers and pulleys can only be determined by measuring the useful and the total work by methods which are explained in books on applied mechanics. A perfect machine, in which these two quantities are equal, has an efficiency of unity; if no work be utilized, the efficiency is zero. One of the aims in constructing a machine is to make its efficiency as near unity as possible, in order that all the work may be utilized. It is, however, rare that efficiencies of over 90 percent can be obtained.

The difference between the total work delivered to a machine and the work utilized by it is called the lost work, and this is lost in overcoming friction. Thus, in raising a weight on an inclined plane, the work lost in friction is $nWl\cos b$. That friction produces heat is a familiar experience, and it has been shown by many investigations that there is an exact equivalence between the amounts of lost work and the amounts of heat generated.

PROBLEMS

271. A force of 25 pounds is applied at one end of a lever 16 feet long. Where must the fulcrum be put so that a pressure of 1000 pounds may be exerted at the other end?

272. A work of 200 foot-pounds is applied to one end of a lever each time that it is depressed. If the efficiency is unity, and the other end is raised 5 inches, what resistance will be then overcome?

273. Solve the last problem if the efficiency is 80 percent.

274. Fig. 101 shows a machine for crushing stone placed at S. The two bars are of equal length and equal inclination to the horizontal, and the applied vertical force P produces large horizontal forces H. If P is 20 pounds and the inclination of the bars is one degree, show that H is 573 pounds.

275. Solve the last problem if the inclination is one minute.

276. Compute the efficiency of a machine which delivers 15.6 horse-powers, after losing 800 foot-pounds of work per second in friction.

277. A screw like Fig. 109 has a pitch of $\frac{1}{5}$ inch, its mean diameter is $3\frac{1}{8}$ inches, and the coefficient of friction is 0.04. Compute its efficiency.

278. If this screw moves a resistance of 1200 pounds through a distance of $2\frac{1}{4}$ inches, what work is lost in friction?

279. In a system of pulleys like Fig. 114 the force P moves through 76 feet and raises 600 pounds through a vertical distance of 6.4 feet. If the efficiency is 0.85, compute the value of P.

280. What horse-power is exerted by this force P when the weight is lifted $10\frac{1}{2}$ inches per second?

CHAPTER VI

GRAVITY AND MOTION

Art. 29. Velocity and Acceleration

Gravity is the name given to the force which causes bodies to press downward upon supports beneath them or to pull downward when they are suspended. When the support is removed from a body, gravity causes it to fall vertically; when a body is thrown upward, gravity causes it to return to the ground. This chapter deals with the motion caused by gravity near the surface of the earth, that is, within the limits of common observation.

Falling bodies, like snowflakes and small rain-drops usually descend in the air with a velocity nearly uniform; but bodies of greater density, like stones and bullets, are observed to increase in velocity as they approach the earth. These two kinds of motion will now be explained.

'Uniform motion' is that in which a body moves through equal distances in equal times. The 'velocity' of the body in this case is the distance passed over in a unit of time; thus if a body in uniform motion passes through a distance of 126 feet in 20 seconds, its velocity is 6.3 feet per second.

'Accelerated motion' is that in which a body moves through a greater distance in each second than in the preceding second. The velocity of the body in this case at any instant is the distance which it would pass through in one second if its motion at that instant continued uniform. Thus, a projectile may be said to have a velocity of 1200 feet per second at a given instant, meaning that it would pass through 1200 feet in one second if its motion at that instant were maintained uniform for one second.

'Uniform accelerated motion' is that in which the velocity of a body increases the same amount in each second. Thus if the velocities of a body are 7, 10, 13, and 16 feet per second at the ends of four consecutive seconds, the velocity has increased 3 feet in each second, and the motion is accelerated uniformly. It is found by observation that the motion of a falling body is of this nature when the resistance of the air is very small.

The unit of time is generally taken as the second, and the unit of distance as the foot or the meter; velocities are then expressed in feet per second or in meters per second. The term 'acceleration' will be used for the increase of velocity in one second, and this is hence expressed in feet per second per second or in meters per second per second. When a body is in uniform motion there is no acceleration; when it is in uniform accelerated motion the acceleration is constant.

For the case of uniform motion, let v be the velocity and l the distance passed over in t seconds; then $v=l/t$ or $l=vt$. This case is graphically represented in Fig. 118, where the time t is measured from A along a horizontal line and the velocity v at any instant is laid off vertically; here the area of the rectangle, or vt, represents the distance passed over.

For the case of uniform accelerated motion let v be the velocity at any instant and f the acceleration. When the body starts from rest, its velocity at the end of the first second is f, at the end of the next second it is $2f$, at the end of the third second it is $3f$, and at the end of t seconds it is tf. This case is graphically represented in Fig. 119, where the time t is measured along a horizontal line and the velocity v at any instant is laid off vertically; here the velocity at the end of any time t is $v=tf$, and the distance which the body moves in t seconds is the area of the triangle, or $l=\frac{1}{2}vt$. When the body has a velocity u at the beginning of the first second, its velocity at the end of that second is $u+f$, and at the end of t seconds it is $v=u+tf$; Fig. 120 shows this case, and the area of the trapezoid represents the distance passed over in t seconds, or $l=\frac{1}{2}(u+v)t$.

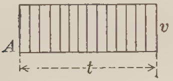

FIG. 118

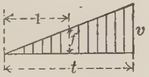

FIG. 119

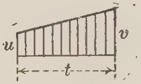

FIG. 120

The above principles render it easy to solve many exercises regarding uniform motion and uniformly accelerated motion. For instance, if a railroad train is moving with a uniform velocity of 88 feet per second, it will in one hour travel a distance of $88 \times 3600 = 316\,800$ feet $= 60$ miles. Again, let a body be moving with a constant acceleration of 12.5 feet per second per second, and let it be observed for 6 seconds; then if it started from rest, its velocity at the end of the time is 75 feet per second; if it had an initial velocity of 33 feet per second at the beginning of the observation, then its velocity at the end of 6 seconds is 108 feet per second.

PROBLEMS

281. A railroad train is in uniform motion at the rate of 44 feet per second. What time is required for it to travel 30 miles?

282. If a train runs at the uniform speed of 60 miles per hour, how many feet will it pass over in one second?

283. A body starting from rest acquires in $4\frac{1}{2}$ seconds a velocity of 150 feet per second. If it is uniformly accelerated, find the acceleration.

284. At the ends of four consecutive seconds the velocities of a body are 31.5, 28.1, 24.7, 21.3 feet per second. What kind of motion is this?

285. What kind of motion is that for which the velocities at the ends of four consecutive seconds are 6.0, 7.1, 8.3, 9.6 feet per second?

286. What distance is passed over by a body in three seconds when it moves uniformly with a velocity of 9.8 meters per second?

287. When the velocity of a body is uniformly accelerated in 0.52 seconds from 0 to 6 feet per second, through what distance has it moved?

288. A body falling from rest has a velocity of 32.2 feet per second at the end of the first second. Show that the distance passed over is 16.1 feet.

289. A body moving with a uniform acceleration of 32.2 feet per second per second is observed to have velocities of 4.3 and 187.5 feet per second at different instants. What time elapsed between these instants?

290. A body moving horizontally at one instant is observed later to be moving obliquely toward the earth. What causes might have produced the change in direction?

ART. 30. VERTICAL FALL OF BODIES

The force which gravity exerts upon any body is ascertained by weighing it with a spring balance. This weight seems the same at the top of a house as on the ground floor, and also the same for all moderate heights above the earth's surface. It has been found, however, that a body weighing 100 pounds at the sea-level weighs about 99.8 pounds on a mountain five miles high. It has also been ascertained that a body weighing 100 pounds at sea-level near the earth's equator weighs about 100.5 pounds at sea-level near the poles. These variations are small and those that can be noted through heights such as can be observed in the case of falling bodies are much smaller. The force of gravity may hence be considered as constant for all common heights of fall.

When a heavy body falls from rest toward the earth, its velocity is observed to increase and to be greatest when it strikes the ground. A body with a large surface, like a feather, may not show this increase in velocity, but when the experiment is tried in a vacuum it is found that a feather and a bullet fall with the same velocity and that this continually increases during the fall. It is hence concluded that the resistance of the air causes the feather to fall slower than the bullet, and that in a vacuum, where this resistance is absent, all bodies fall with the same accelerated velocity. In the following pages the motion of bodies under the action of gravity is supposed to occur in a vacuum.

Since the force of gravity is constant for all common heights of fall, it follows from Axiom 4 of Art. 2 that the effect produced upon the velocity is the same in each second, or that the acceleration is constant. The fall of the body hence occurs with uniform accelerated motion (Art. 29). This conclusion has been verified by many experiments and measurements, and it is found that when a body falls vertically the mean value of its acceleration in the United States of America is 32.16 feet per second per second; that is, the gain in velocity is 32.16 feet in each second, whatever the velocity may be at the beginning of that second. The letter g will be used for this acceleration.

The equations deduced in the last article for uniformly accelerated motion hence apply directly to the case of a body falling vertically. Let the body start from rest at the beginning of the time t, and let g be the acceleration or gain per second in velocity. The velocity v at the end of the time t is then $v=gt$. The height h through which the body falls in this time is $h=\frac{1}{2}vt$; or inserting in this the above value of v, the height of fall is $h=\frac{1}{2}gt^2$. Also inserting the value $t=v/g$ in $h=\frac{1}{2}vt$, there results $h=v^2/2g$. Accordingly,

$$v=gt \qquad h=\tfrac{1}{2}gt^2 \qquad v^2=2gh$$

are three important formulas for a body falling vertically from rest, the resistance of the air being neglected.

Using 32.16 feet per second as a mean value of the acceleration g, the following values of the velocity acquired and of the distance fallen at the end of several seconds are computed:

For time $t=$	0,	1,	2,	3,	4	seconds
Velocity $v=$	0,	32.2,	64.3,	96.5,	128.6	feet per second
Height $h=$	0,	16.1,	64.3,	144.7,	257.3	feet

These results have been confirmed by experiments conducted in such a manner that the resistance of the air is eliminated.

The above equations contain the constant g and the three quantities h, v, t. When t is given the velocity may be found from $v = gt$, and the height from $h = \frac{1}{2}gt^2$, as is done above. These equations show that the velocity is proportional to the time, and that the height of fall is proportional to the square of the time. Thus, if the time is doubled, the velocity becomes twice as great, but the height of fall becomes four times as great.

The formula $h = \frac{1}{2}gt^2$ may be used to find the time required for a body to fall through the vertical height h. For example, let the height of a tower be 199 feet; then a stone dropped at the top will fall to the ground in $t = (2 \times 199/32.16)^{\frac{1}{2}} = 3.52$ seconds, if not retarded by the resistance of the air.

The third formula, $v^2 = 2gh$, may be used to find the velocity when the height is given, or the height when the velocity is given. For example, if it be required to find the velocity which a body will acquire in falling through a height of 199 feet, then $v = (2 \times 32.16 \times 199)^{\frac{1}{2}} = 113.2$ feet per second, if there is no resistance of the air. This formula shows that the velocity is proportional to the square root of the height, or that the height is proportional to the square of the velocity.

Owing to the friction of the air, the actual velocities and heights of a falling body for a given time t are always less than those computed from the above formulas, and the actual time for a given height is greater. The amount of retardation depends upon the surface of the body as compared with its density. Spherical stones and bullets suffer but little retardation when the height of fall is less than 300 feet. It may be noted here also that the value of the acceleration g may vary between 32.26 and 32.06 feet per second per second, the former value being for sea-level at the poles of the earth, and the latter for high mountains at the equator. The value 32.16 feet per second per second is generally used in the United States of America in engineering computations. In Arts. 35 and 40 methods for determining the value of g will be explained.

PROBLEMS

291. A body falling vertically has a velocity of 150 feet per second. How many seconds have elapsed since it started from rest?

292. Through what height will a body fall in one-tenth of a second? in two-tenths of a second?

293. What time is required for a body to fall one inch? also four inches?

294. What velocity is acquired by a body in falling through a height of 0.01 feet? also for 0.09 feet?

295. Show that the distances through which a body falls in the first, second, third, and fourth seconds are equal to $\frac{1}{2}g$, $\frac{3}{2}g$, $\frac{5}{2}g$, and $\frac{7}{2}g$.

296. A lead bullet is observed to fall from rest through a height of 78 feet in $2\frac{1}{5}$ seconds. Compute the value of the acceleration.

297. A stone is dropped from the top of a tower and 2 seconds later another stone is dropped from a point half-way up the tower. If they strike the ground at the same time, find the height of the tower.

298. A falling body passes through a distance of 215 feet in the last second of its fall. Find the height from which it fell and the total time of its fall.

299. The mean value of g in the metric system is 9.80 meters per second per second. Compute the velocities and heights for a falling body at the end of 1, 2, 3, and 4 seconds.

300. When the velocity is 1.37 meters per second, through how many centimeters has the body fallen?

ART. 31. OBLIQUE FALL

When a body is placed upon an inclined plane, it will slide down that plane if the frictional resistance does not prevent such motion. Imagine a plane perfectly smooth, so that this resistance is absent; let a body of weight W be upon this inclined plane BC, which is inclined to the horizontal at the angle b; then the force P which acts parallel to the plane is $W \sin b$, and this is the only force which causes motion, for the normal component $W \cos a$ can have no influence in the direction BC (Art. 2, Axiom 5). The force W is able to produce an acceleration g in a vertical direction, but the force $W \sin b$ can produce only the acceleration $g \sin b$ in the direction BC, for the effects of forces are proportional to the magnitudes of the forces (Art. 2, Axiom 4).

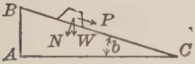

FIG. 121

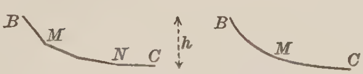

FIG. 122

A body sliding down an inclined plane of complete smoothness hence has the constant acceleration $g \sin b$, and its motion is uniformly accelerated. At the end of one second after starting from rest, its velocity is $g \sin b$, at the end of t seconds its velocity is $v=(g \sin b)t$. Let l be the distance, measured along BC, which the body passes over in t seconds; this distance, as shown in Art. 29, is $l=\frac{1}{2}vt$. Eliminating t from these two equations, there results $l=v^2/2g \sin b$. Accordingly,

$$v=(g \sin b)t \qquad l=\tfrac{1}{2}(g \sin b)t^2 \qquad v^2=2(g \sin b)l$$

are the equations for the fall of a body on an inclined plane, if there is no resistance of the air and no friction on the plane. These are the same as the equations for vertical fall, except that g is replaced by $g \sin b$, and the distance is measured along the plane instead of vertically. The velocity acquired in a given time is hence less than under vertical fall; for example, when b is 30 degrees, $\sin b$ is $\frac{1}{2}$, and the velocity is only one-half of that for a vertical fall.

Let h be the vertical height corresponding to the length l on the inclined plane; then $l \sin b$ equals h, and hence the third equation above becomes $v^2=2gh$. This proves that the velocity acquired by a body in falling through the vertical height h is the same whether the fall be in a vertical line or along an inclined plane. The time occupied in the fall from a higher to a lower level is, however, least when the fall is vertical.

The resistance of friction along the inclined plane may be taken into account as follows: Let n be the coefficient of friction (Art. 19) and W the weight of the body; then $nW \cos b$ is the resisting force of friction, and the resultant force parallel to the plane which causes motion down the plane is $W(\sin b-n \cos b)$. Now if W can acquire the acceleration g when falling vertically, it can only acquire the acceleration $g(\sin b-n \cos b)$ when sliding down the plane. The above formulas for motion on a frictionless plane will hence apply to an actual plane if $\sin b$ be replaced by $\sin b - n\cos b$. For example, if b be 30° and n be 0.5, then the acceleration is $32.16(0.5 - 0.433) = 2.15$ feet per second per second, and the distance passed over in the first second will be 1.07 feet, or about one-sixteenth of that which occurs in vertical fall. When n and b have such values that $\sin b - n\cos b$ is zero, the body remains at rest upon the plane.

When a body slides down a succession of inclined planes BM, MN, etc., as in Fig. 122, and h is the total vertical fall from B to C, the formula $v^2 = 2gh$ also applies if there is no friction and no resistances due to the angles M, N, etc. For the velocity at M depends only upon the vertical height from B to M, and if this is not altered at M, the velocity at N is due to the vertical fall from B to N. When these planes become very short, they form the curve BMC, and the velocity of a body sliding down this curve is also given by $v^2 = 2gh$, where h is the vertical height of B above C. The time is, however, different from that of a vertical fall or from that of the fall down an inclined plane. There is a certain curve joining the points B and C for which the time of fall is less than for any other path between these points, and by the help of a branch of mathematics called the calculus of variations it can be shown that this curve is a cycloid.

PROBLEMS

301. An inclined plane has a rise of 30 feet and a base of 40 feet. Find the time required for a body to slide down the hypothenuse if it be perfectly smooth.

302. Find the time for the last problem if the coefficient of friction is 0.05; also when it is 0.50.

303. Compute the velocity at the foot of the above plane

(*a*) when it is perfectly smooth, (*b*) when the coefficient of friction is 0.50.

304. Show that the velocity of a body at the foot of a rough inclined plane of vertical height h is given by $v^2 = 2gh(1 - n \cot b)$.

305. A plane has an inclination of 40° 15′ and its length is 40 feet. Find the velocity of a body at the foot, if it starts from rest at the top and there are no resistances.

306. A body starts from rest at the top of an inclined plane and reaches the foot in 6.4 seconds. If the length of the plane is 100 feet and it is perfectly smooth, compute its inclination.

307. A body slides down a smooth cycloidal curve connecting the points B and C (Fig. 122). If the vertical height of B above C is one foot, find the velocity at C.

308. Let the horizontal distance between B and C in the last problem be 20 feet. Show that the time of fall is more than two seconds.

309. In a vertical circle let chords be drawn from the lowest point. Show that the time occupied by a body in sliding down any chord is equal to the time of falling through the vertical diameter.

310. A stone is dropped from the top of a tower and at the same instant another stone is dropped from a point half-way up the tower. If the first stone strikes the ground 1.2 seconds later than the second, compute the height of the tower.

ART. 32. POTENTIAL AND KINETIC ENERGY

In Art. 21 work was defined as resistance multiplied by the distance through which it is overcome, and the work done in lifting a body against the constant resistance of gravity was expressed by Wh, where W is the weight of the body and h is the vertical height of lift. When this body falls through the height h, gravity is said to store the work Wh in it, and Fig. 90 shows how this work can be utilized in lifting the body to the level from which it started, if no work is lost in frictional resistances. It may hence be said that a body of weight W, situated at the vertical height h above the surface of the earth, has a capacity to do the work Wh.

'Energy' is the capacity to perform work, and a body of weight W at the vertical height h above the surface of the earth is said to have the potential energy Wh with respect to the earth, because it can be made to do this amount of useful work if frictional resistances are eliminated. 'Potential energy' is the work which a body can perform by virtue of its position above the earth's surface and, like work, it is expressed in foot-pounds. Thus, an iron ball weighing 50 pounds and situated 4 feet above the earth's surface has a potential energy of 200 foot-pounds, that is, it has the capacity to perform 200 foot-pounds of work. If this ball strikes the head of a nail and drives it 1.5 inches into a plank against a constant resistance F, the mean value of this resistance is $F=200\times 12/1.5=1600$ pounds.

Fig. 122 shows how a body may be guided in its fall so that the direction of its velocity becomes horizontal as it reaches the earth, and in the last article it is proved that this velocity depends only upon the vertical height h, and that in the absence of resistances the equation $v^2=2gh$ is always true. Hence follow the formulas

$$h=\frac{v^2}{2g} \qquad \text{and} \qquad Wh=W\frac{v^2}{2g}$$

in which Wh is the potential energy of the body in its position of rest at the upper level, while $W.\,v^2/2g$ is the energy of the body at its lower level, that is, by virtue of the velocity v it has a capacity to perform the work $W.\,v^2/2g$. The height h is that between the initial and final positions of the center of gravity of the body.

'Kinetic energy' is the work which a body can perform by virtue of its velocity. The direction of the velocity is immaterial, for the equation $v^2 = 2gh$ holds whether the velocity be vertical, inclined, or horizontal at the end of the height h, as Arts. 30 and 31 show. Whenever a body of weight W is moving with a velocity v in any direction with respect to the earth, it is said to have a kinetic energy of $W.\,v^2/2g$ with respect to the earth, g being the acceleration of the body in a vertical fall. For example, let a cannon-ball weighing 305 pounds be moving with a velocity of 1500 feet per second as it leaves the muzzle, then its kinetic energy is $305 \times 1500^2/64.32 = 10\,670\,000$ foot-pounds, that is, this moving ball would be able to lift a weight of 10 670 pounds through a vertical height of 1000 feet if no work is lost in frictional resistances.

When a body is falling through the height h the sum of the potential and kinetic energies at any instant is constant and equal to Wh. To show this, let x be any vertical height measured upward from the earth, and let v_x be the velocity which the body has acquired in falling through the vertical height $h - x$. Then the relation between v_x and $h - x$ is given by $v_x^2 = 2g(h - x)$. Accordingly it follows that

$$x + \frac{v_x^2}{2g} = h \qquad \text{or} \qquad Wx + W\frac{v_x^2}{2g} = Wh$$

in which Wx is the potential energy and $W.v_x^2/2g$ is the kinetic energy of the body at the height x, while Wh is the potential energy of the body before it began to fall. This law is true whatever may be the direction of the velocity v_x, that is, the body may be falling vertically or obliquely. For example, let a body weighing 25 pounds start from rest at B (Fig. 122) and slide down the frictionless curve BMC to the earth at C. Let M be 5 feet and C be 12 feet vertically below B. Then the potential energy of the body at B with respect to the earth is 300 foot-pounds, while its potential energy at M is 175 foot-pounds; accordingly its kinetic energy at M is 125 foot-pounds. At B the energy of the body is entirely potential, at any point between B and C its energy is partly potential and partly kinetic, and on arriving at C its energy is entirely kinetic.

When a body of weight W is in motion in any direction with the velocity u at a vertical height h above the earth's surface, its energy with respect to the earth is partly kinetic and partly potential. Its kinetic energy is $W.u^2/2g$ and its potential energy is Wh. When the body reaches the earth it has a certain velocity v, and its kinetic energy is $W.v^2/2g$. If no work has been lost in friction, its kinetic energy as it strikes the earth must be equal to the sum of the kinetic and potential energies which it had when at the height h; accordingly, $W.v^2/2g$ equals $W.(u^2/2g+h)$. For example, if a cannon-ball weighing 305 pounds is fired with a horizontal velocity of 800 feet per second from an elevation of 480 feet above a target, its kinetic energy with respect to the target is $305 \times 800^2/64.32 = 3\,035\,000$ foot-pounds and its potential energy is $305 \times 480 = 146\,400$ foot-pounds.

Hence, if there is no work lost in overcoming the resistance of the air, the kinetic energy of the ball, as it strikes the target, is 3 181 400 foot-pounds.

PROBLEMS

311. When a mine cage weighing 6500 pounds is at the top of a vertical shaft 185 feet deep, find its potential energy with respect to the bottom of the shaft.

312. If the suspending rope of this cage breaks and the cage reaches the bottom of the shaft with a velocity of 96 feet per second, find its kinetic energy.

313. If this moving cage is stopped in five seconds, what horse-power does it have during that time?

314. Show that the kinetic energy of a moving body will be quadrupled if its velocity is doubled.

315. When a body falls from a height h, without air resistance, show that its kinetic energy at the end of 3 seconds is nine times as great as at the end of the first second.

316. A body weighing 100 pounds moves down an inclined plane in 7.4 seconds, the inclination being $27° 15'$ and the coefficient of friction being 0.39. Find its kinetic energy when it reaches the foot of the plane.

317. A body weighing 400 pounds is moving horizontally with a velocity of 87 feet per second. To what height can it lift a body weighing 50 pounds?

318. A body weighing 400 pounds is moving vertically upward with a velocity of 87 feet per second. To what height will it rise ?

319. A constant force acting through a distance of 3 inches on a body of 8 pounds weight imparts to it a velocity of 4.35 feet per second. Find the magnitude of the constant force.

320. A body weighing 75 pounds, moving with a velocity of 100 feet per second, meets a constant resistance of 150 pounds, which continues for a distance of 30 feet. What is the

kinetic energy of the body and its velocity after passing this resistance?

ART. 33. MOTION OF PROJECTILES

When an applied force has acted upon a free body of weight W and then ceased, it moves in the direction of that force with a certain velocity u. Such a body is called a 'projectile,' and it is required to find the circumstances of its subsequent motion under the action of gravity, for gravity causes it ultimately to fall to the surface of the earth.

The simplest case is that when the applied force produces an initial vertical downward velocity u, as might occur for a bullet fired from a gun in a balloon. The vertical acceleration, or gain in velocity per second due to the force of gravity, being g, its velocity at the end of the first second is $u+g$, and at the end of t seconds the velocity $v=u+gt$. The distance fallen through in t seconds is then $h=\frac{1}{2}(u+v)t$, as in all cases of uniformly accelerated motion with initial velocity (Art. 29). Taking the value of v from the first equation and inserting it in the second gives an equation between the height h and the time t; taking the value of t from the first equation and inserting it in the second gives an equation between the height h and the velocity v. Accordingly

$$v=u+gt \qquad h=ut+\tfrac{1}{2}gt^2 \qquad v^2=u^2+2gh$$

are three important formulas for the case of vertical fall with a given initial velocity, air resistance being neglected.

When the body is projected vertically upward, the velocity at the end of the first second is $u-g$, since gravity now diminishes the upward initial velocity u; the upward velocity at the end of t seconds is then $v=u-gt$, so that v decreases with the time and finally becomes zero, at which instant the body reaches the highest possible elevation. As before the distance d passed over in the time t is $d=\frac{1}{2}(u+v)t$, and

$$v=u-gt \qquad d=ut-\tfrac{1}{2}gt^2 \qquad v^2=u^2-2gd$$

are the three formulas for a body projected vertically upward, these being similar to those for the previous case except that all terms containing g are reversed in sign. This is uniformly retarded motion, since v is always less than u. The first equation shows that the velocity v becomes zero when $t=u/g$; this value inserted in the second equation gives $d=u^2/2g$, which is the height to which the body will rise, and the same result is found by making v equal to zero in the third equation. The upward vertical velocity $(2gd)^{\frac{1}{2}}$ must hence be given to a body in order that it may rise to the height d and then stop; it remains at rest, however, but an instant, for the force of gravity causes it immediately to begin to fall.

When a body is projected in any direction with a velocity u from a point whose height above the earth's surface is h, its velocity v as it strikes the earth may be deduced by equating its final kinetic energy to the sum of its potential and kinetic energies in its initial position. The final kinetic energy is $W.v^2/2g$, the initial kinetic energy was $W.u^2/2g$, and the initial potential energy was Wh; therefore

$$W\frac{v^2}{2g}=W\frac{u^2}{2g}+Wh \quad \text{or} \quad v^2=u^2+2gh$$

This is the third equation deduced above for vertical fall, and it is independent of the direction of the initial velocity u. The time of flight in the curved path is, however, always greater than that for vertical fall. Fig. 123 shows three bodies projected in three different directions with the same initial velocity u from points at the vertical height h above the ground. The paths described are quite different, but the velocity v on reaching the earth at C is the same in the three cases and is given by $v^2 = u^2 + 2gh$; the time required to reach the earth is evidently the greatest when the direction of u has an upward inclination.

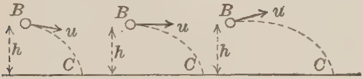

FIG. 123

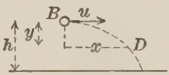

FIG. 124

The curve described by a projectile when the initial velocity u is not vertical is a parabola, that is, a curve such that the vertical projection of any portion below its highest point is proportional to the square of its horizontal projection. This will be shown for Fig. 124, where the initial velocity u is horizontal. Let D be any point on the curve, y the vertical and x the horizontal projection of the curve BD, and t the time required to pass from B to D. If it were not for the initial velocity, the body would fall vertically through the distance y in the time $t = (2y/g)^{\frac{1}{2}}$, as is known by Art. 30. If it were not for the force of gravity, the body would move horizontally with the uniform velocity u and pass over the distance x in the time $t = x/u$. Equating these two values of t, there is found $2u^2y = gx^2$, so that values of

y are proportional to the squares of x, and hence the curve is a parabola. When y becomes equal to h, the horizontal distance x is called the range of the projectile on the ground, and its value is $(2u^2h/g)^{\frac{1}{2}}$. For example, if a bullet be fired horizontally with a velocity of 430 feet per second from a point 60 feet above the earth, its range is 830 feet. Owing to air resistance, however, the actual range will be somewhat less than this computed value.

By the help of the above formulas numerous problems relating to the motion of projectiles may be solved. For instance, let a body be projected vertically upward with a velocity of 150 feet per second, and let it be required to find the time in which it would rise a distance of 300 feet. Here the equation $d = ut - \frac{1}{2}gt^2$ applies; making $d = 300$ feet, $u = 150$ feet per second, and solving the quadratic, there is found $t = 2.88$ or $t = 6.42$ seconds; the first of these values gives the time when the body is passing upward through a point 300 feet above the earth, and the second the time when it is at the same point in its fall from the highest elevation which it attains. Air resistance is neglected here and in the following problems.

PROBLEMS

321. A body is projected vertically downward with a velocity of 125 feet per second from a point 300 feet above the earth. Find the time occupied in the fall.

322. Using the above data and the time, compute in two ways the velocity of the body when it strikes the earth.

323. From the foot of a tower a body is projected upward with a velocity of 87 feet per second, and at the same instant

another body is let fall from the top of the tower. If they meet half-way up the tower, find its height.

324. A body enters a room through the ceiling with an initial velocity of 65 feet per second and making an angle of 30 degrees with the vertical. Compute its velocity when it strikes the floor, the height of the room being 12 feet 3 inches.

325. A body is projected vertically upward with a velocity of 83 feet per second. What time elapses before it returns to earth? What is its velocity when it is at a height of 110 feet?

326. A ball is fired horizontally from a height of 5.7 feet above the ground and strikes the ground at a horizontal distance of 1000 feet from the muzzle of the gun. Compute the initial velocity.

327. The guns of a fort are 157 feet above the water level of a harbor. What range do they command, if the horizontal initial velocity of the projectiles is 2000 feet per second?

328. If the direction of the initial velocity in the last problem has an upward inclination of 45 degrees, show that the range will be very much increased.

329. The kinetic energy of a ball as it leaves a gun is 4 000 000 foot-pounds. If the initial velocity is 500 feet per second, compute the weight of the ball. If the elevation of the gun is 95 feet higher than the vessel, compute the final velocity of the ball.

330. A poet wrote: "Swift of foot was Hiawatha; he could shoot an arrow from him and run forward with such fleetness that the arrow fell behind him. Strong of arm was Hiawatha; he could shoot ten arrows upward, shoot them with such strength and swiftness that the tenth had left the bowstring ere the first to earth had fallen." Compute the running speed of Hiawatha in miles per hour, assuming that he could shoot ninety arrows per minute and that when shooting forward he aimed at an angle of 45 degrees with the horizontal.

ART. 34. COMPOSITION OF VELOCITIES

When a body is moving with the velocity v, its velocity in any direction is defined to be the component of v in that direction. If a velocity v in a horizontal plane makes an angle a to the eastward of a meridian line, its northward velocity is $v \cos a$ and its eastward velocity is $v \sin a$. If the velocity v is inclined at an angle a to the horizontal, its horizontal velocity is $v \cos a$ and its vertical velocity is $v \sin a$.

In the last article it was taken for granted that the body in Fig. 124 would continue to move with the initial velocity u if it were not for the action of gravity. This assumption is justified by all experience, and the following axioms may be stated as truths of experience:

> Axiom 9. A body in uniform motion does not change the direction or magnitude of its velocity unless force be applied to it.
>
> Axiom 10. When a body in uniform motion has been deflected from its original direction by gravity only, its velocity in a horizontal direction remains unaltered.

The first of these axioms is contained in the laws of motion announced by Newton in 1687, and he notes in support of the same that projectiles maintain their velocities unless retarded by the resistance of the air or drawn downward by gravity, and that a rotating top does not lose its velocity except through air resistance and friction. Many other facts may be cited and no case has ever been observed in which the axiom is violated. Axiom 10 is really a corollary from Axioms 1 and 9, but many facts may be mentioned in its support, one of which is that a body dropped from a moving car continues to move horizontally with the velocity of the car except in so far as retarded by air resistance.

The first paragraph of this article indicates that a velocity may be replaced by two components. Conversely, if the components be given, the velocity itself may be found. Thus, let the body B in Fig. 125 be projected with the initial velocity u from a point at the height h above the ground, this velocity making an angle a with the horizontal. The horizontal velocity of the body at C is $u \cos a$, since this is unaltered by gravity. The vertical velocity at B is $u \sin a$, and that at C is $u \sin a + gt$, since gravity causes the acceleration g in each second. The resultant velocity at C is represented by the diagonal of a rectangle having the horizontal and vertical velocities for its sides, and accordingly

$$v^2 = (u \cos a)^2 + (u \sin a + gt)^2 = u^2 + 2g(ut \sin a + \tfrac{1}{2}gt^2)$$

Now, the quantity $ut \sin a + \frac{1}{2}gt^2$ is the height h, and hence $v^2 = u^2 + 2gh$, which is the same result as that deduced in Art. 33.

The above considerations indicate that the resultant of two component velocities is the diagonal of a parallelogram formed upon sides representing those components, and this may also be inferred from the parallelogram of forces, since each force might produce a velocity in its direction, and the actual velocity is that produced by the resultant of the two forces.

'Absolute velocity' is that with respect to the earth, this being regarded as at rest. All the velocities thus far considered have been absolute. 'Relative velocity' is that with respect to a moving body; for example, if a person on a moving car throws a stone, it has a certain relative velocity with respect to the car and another absolute velocity with respect to the earth. This absolute velocity may be also found from the parallelogram of velocities with the help of Axiom 10.

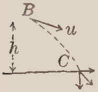

FIG. 125

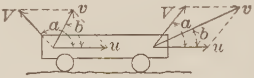

FIG. 126

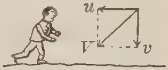

FIG. 127

Let Fig. 126 represent a car moving from left to right with the absolute velocity u, and let a person on the car throw a stone with the velocity V relative to the car in a direction making the angle a with the direction of u. If u were the only velocity, the stone would move the distance u to the right in one second; if V were the only velocity, it would move the distance V in one second in the direction of the throw. Hence the resultant velocity v is the diagonal of the parallelogram which is formed upon u and V as sides, and

$$v^2 = u^2 + V^2 + 2uV \cos a \qquad \sin b = (V/v) \sin a$$

give the magnitude and direction of this absolute velocity. If $a = 0°$, then $v = u + V$, and $b = 0°$; if $a = 180°$, then $v = u - V$ and $b = 180°$; if $a = 90°$, then $v = (u^2 + V^2)^{\frac{1}{2}}$ and $\sin b = V/(u^2 + V^2)^{\frac{1}{2}}$. These are the magnitudes and directions of the absolute velocities at the instant of the throw.

Another instance of the combination of absolute and relative velocities is shown in Fig. 127, where a man is seen running from left to right with the absolute velocity u, while rain is falling vertically with the absolute velocity v, and it is required to find the velocity of the rain-drops relative to the man. To do this the man should be considered as at rest, so that the earth and the rain-drops may be regarded as moving from right to left with the relative velocity u. The resultant of u and v then is $V=(u^2+v^2)^{\frac{1}{2}}$, and the angle which it makes with the vertical is given by $\tan b = u/v$. Thus, if the man runs with a velocity equal to that of the falling rain, the rain strikes him in the face with a relative velocity of $V=1.414u$, and its direction makes an angle of 45 degrees with the vertical.

PROBLEMS

331. A man can row 2.1 miles per hour in still water. How long will it take him to cross a pond 1320 feet wide? How long will it take him to cross a stream of the same width which is flowing with a velocity of 1.5 miles per hour?

332. A boat is moving north with a velocity of 8 miles per hour and another boat is moving east with a velocity of 12 miles per hour. Find the velocity of the first boat relative to the second.

333. A steamship is moving west with a velocity of 22 miles per hour, and to the passengers the wind appears to blow from the north with a velocity of 15 miles per hour. Find the absolute velocity and direction of the wind.

334. A wagon is moving with the velocity v. Find the absolute velocity of the top of a wagon wheel.

335. A body is projected horizontally with a velocity of 200 feet per second. Find the magnitude and direction of its velocity after it has been in motion for three seconds.

336. A body is projected with a velocity of 301.6 feet per

second in a direction making an angle of 60° with the vertical. How many seconds will elapse before it reaches its highest elevation?

337. A man traveling east at the rate of 3 miles per hour notes that the wind appears to blow directly from the north; on doubling his speed it appears to blow directly from the northeast. Find the true direction and velocity of the wind.

338. A bullet is fired vertically downward from a car which is moving horizontally at a speed of 88 feet per second. If the velocity of the bullet is 200 feet per second relative to the car, find its absolute velocity.

339. If the bullet of the last problem weighs 0.23 pounds, find its kinetic energy with respect to the earth. If the muzzle of the gun is 12 feet above the earth, find the kinetic energy of the bullet on reaching the earth.

340. A person on the rear of a moving car throws a stone backward with a velocity of 27 feet per second, and in a direction making an angle of 30 degrees with the vertical. To a person standing on the earth this stone appears to drop in an exact vertical. Find the speed of the car in miles per hour.

CHAPTER VII

INERTIA AND ROTATION

ART. 35. INERTIA OF MATTER

When a body at rest is put into motion by an applied force, a resistance is experienced; when a body in motion is brought to rest by an applied force, a similar resistance is experienced. This resistance is called 'inertia,' and it is found to be the greater the more quickly the change of velocity is made. Inertia thus depends not on the velocity but on the change of velocity, that is, upon the gain in velocity when the body is put into motion, and upon the loss of velocity when its motion is destroyed.

In Art. 30 it was shown that a constant gain in the velocity of a free body occurs under the constant force of gravity, and from Axiom 4 of Art. 2 it also follows that any constant force acting on a free body produces a constant acceleration. Art. 33 shows that a body projected vertically upward suffers a constant loss in velocity from the resistance of gravity, and Axiom 4 indicates that any force constantly acting to stop a moving body produces a constant retardation. This conclusion is verified by all experience, and it may hence be laid down as a law of nature that constant forces acting upon a free body produce accelerations proportional to their magnitudes.

Let W be the weight of any body at the place where the acceleration of gravity is g. Let a force F act in any direction upon this free body, and f be the acceleration which results. Then, from the above law,

$$F/W=f/g \quad \text{or} \quad F=W.f/g$$

This value of F is the constant force which is required to cause the acceleration f in the body of weight W, or the force required to overcome the resistance of inertia. Hence inertia is proportional to the weight of the body and to the acceleration or retardation which the body undergoes, that is, to the gain or loss per second in its velocity.

For example, take a body of 100 pounds weight resting upon a frictionless horizontal plane, and let it be required to find what constant horizontal force will produce an acceleration of 3 feet per second per second. The equation gives $F=100\times3/32.16=9.33$ pounds. If this force be applied to the body for 5 seconds, it will attain a velocity of $5\times3=15$ feet per second. If it then be required to stop the body in 2 seconds, the retardation in its velocity must be $7\frac{1}{2}$ feet per second, and the constant force needed to do this will be $F=100\times7\frac{1}{2}/32.16$ $=23.32$ pounds; this is equal to the resisting force of inertia of the body.

When a body is projected vertically upward by the action of a constant force P, this must overcome both the weight of the body and the resistance of inertia; hence the value of this constant force is

$$P=W+W.f/g \qquad \text{or} \qquad P=W(g+f)/g$$

In this equation W may represent the weight of a man and P the upward pressure exerted upon him by the floor of an elevator which moves upward with the acceleration f, and it is seen that P is always greater than W. When the elevator moves downward, f becomes negative and P is less than W. When the elevator moves with uniform velocity, either upward or downward, f is zero and P equals W.

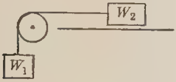

FIG. 128

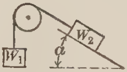

FIG. 129

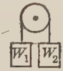

FIG. 130

The inertia formula may be used to compute the acceleration f of bodies so connected that they must always move with the same speed. For example, let the two weights W_1 and W_2 in the following figures be connected by a flexible cord passing over a pulley, the weight W_1 moving down. In Fig. 128 the constant moving force F is W_1 and the weight W_1+W_2 moves with the acceleration f, if there is no friction and air resistance; accordingly,

$$f=g\,.\,F/W \qquad \text{or} \qquad f=g\,.\,W_1/(W_1+W_2)$$

and f will be small when W_1 is small and W_2 large. For Fig. 130 the weight W_1 is made a little heavier than W_2, so that the constant moving force is W_1-W_2 and the weight moved is W_1+W_2; hence

$$f=g\frac{W_1-W_2}{W_1+W_2} \qquad \text{or} \qquad g=f\frac{W_1+W_2}{W_1-W_2}$$

from which f may be found when g is given, or g be computed when f has been observed. This is the principle of Atwood's machine for determining g. In this apparatus the cord and pulley are made very light and W_1 is but little heavier than W_2, so that the motion is slow and the time t required for W_1 to fall a distance l can be accurately observed. Then the value of f is found from $2l/t^2$, and from this and the known weights the value of the acceleration g can be computed.

PROBLEMS

341. When a man stands in a rapidly moving car and it is suddenly brought to rest, what happens to him? When a man steps off a moving car, in which direction should he face, and why?

342. A constant force of 80 pounds acting on a body for one second causes its velocity to increase from 15 to 17 feet per second. Compute the weight of the body.

343. Two bodies are acted upon by the same constant force and the acceleration of one is observed to be three times that of the other. Compare their weights.

344. A body moving with a velocity of 27 feet per second is brought to rest in three seconds by an opposing constant force of 100 pounds. Find the weight of the body.

345. A man stands upon a weighing-machine on the floor of an elevator at rest and it indicates 163 pounds. How much will it indicate when the elevator rises with uniformly accelerated motion through the height of 39 feet in 3.4 seconds?

346. How much will it indicate if the rope of the elevator breaks and the elevator falls freely under gravity?

347. For Fig. 129 deduce a formula giving the value of f in terms of W_1, W_2, and the angle of inclination a.

348. For Fig. 129 let $W_1 = 40$ pounds and $a = 16° 15'$. Neglecting all resistances, find the value of W_2 so that the acceleration of the two weights may be 4.04 feet per second per second.

349. For Fig. 128 show that the tension in the cord is $W_2 \cdot f/g$ and that this may be written $W_1 W_2/(W_1 + W_2)$.

350. Two weights of 13 and 12 pounds are arranged as

in Fig. 130. Find their common velocity at the end of three seconds after starting from rest. If the cord then breaks, how much higher will the smaller weight rise?

ART. 36. INERTIA AND ENERGY

The constant force F required to overcome the inertia of a body and cause it to have an acceleration f in the direction of the force is $F=W.f/g$ as shown in Art. 35. If the body starts from rest, the velocity at the end of one second is f and at the end of t seconds it is $v=ft$. Any body moving with the velocity v may hence be regarded as having acquired its velocity from some constant force F which has acted for t seconds and then ceased, the acceleration having been $f=v/t$. Inserting this value of f in the inertia formula, it becomes

$$F=W.v/gt \qquad \text{or} \qquad Ft=W.v/g$$

from which F may be computed when v and t are given. This constant force F applied to oppose the motion of the body will also bring it to rest in the time t. For example, take a railroad train, weighing 650 000 pounds, which is moving with a speed of 60 miles per hour or 88 feet per second, and let it be required to find what constantly acting force will bring it to rest in 30 seconds. Here the computation gives $F=59\,300$ pounds, and this force may be applied by causing brakes to act upon the wheels.

In all cases of uniformly accelerated motion, when the body starts from rest and acquires the velocity v in the time t, the distance l passed over is $l=\frac{1}{2}vt$ (Art. 29), or the time t is $2l/v$. Substituting this value of t in the above equations, they become

$$F = W.v^2/2gl \qquad \text{or} \qquad Fl = W.v^2/2g.$$

The first of these equations gives the constant force F which must be exerted on a body in order that it may acquire the velocity v after traveling the distance l. The second gives the work Fl which this constant force performs, and this is found to be equal to the kinetic energy of the moving body (Art. 32). The result of a constant force in overcoming the inertia of a body through a certain distance and giving it a final velocity v is therefore to store up kinetic energy in it. This has been already shown in Art. 32 for the force of gravity, and it is now seen to be true for any constant force.

A body having the kinetic energy $Wv^2./2g$ can hence be brought to rest by a constant opposing force F acting through a certain distance l. For example, take a railroad train weighing 650 000 pounds and moving with a velocity of 88 feet per second, and let it be required to find the distance it will travel before coming to rest when opposed by a constant force of 59 300 pounds. The kinetic energy of the train is $650\,000 \times 88^2/64.32 = 78\,259\,000$ foot-pounds, and then $l = 78\,259\,000/59\,300 = 1320$ feet.

When a body is in uniform motion with the velocity u and a constant force F acts upon it in any direction, producing the acceleration f in that direction, the velocity V acquired in that direction at the end of the time t is $V = ft$, and hence it follows that the equations

$$F = W.V/gt \qquad \text{or} \qquad Ft = W.V/g$$

give the force F required to overcome the inertia. These are the same as for the case of a body starting from rest, except that V is the velocity relative to the body. For example, let a body weighing 10 pounds be moving at the rate of 75 feet per second, and let a constant force act upon it for 3 seconds in a direction normal to that of u and produce the velocity of 50 feet per second in that direction; then this constant force $F = 10 \times 50/32.16 \times 3 = 5.18$ pounds. From Art. 34 the velocity v of the body at the end of the given time is $v = (75^2 + 50^2)^{\frac{1}{2}} = 90.1$ feet per second.

When a body is moving with the velocity u, let a constant force F act upon it in any direction through the distance l, overcoming the resistance of inertia and increasing the velocity to v. The initial kinetic energy is $W.u^2/2g$, the work performed by the constant force is Fl, and the final kinetic energy is $W.v^2/2g$. Accordingly if there be no work lost in friction or heat,

$$W\frac{u^2}{2g} + Fl = W\frac{v^2}{2g} \qquad \text{or} \qquad Fl = W\frac{v^2 - u^2}{2g}$$

The last equation gives the work Fl required to increase the velocity of the body from u to v, whether the direction of it coincides with that of u or is inclined to it. It also gives the work Fl required to decrease the velocity of the body from v to u, but in this case the opposing force must necessarily act in the reverse direction of u; thus when a train runs around a curve there is a lateral pressure upon it from the outer rail, but this does no work because it acts through no distance in its direction; it produces, however, a resisting friction F which acts in the opposite direction to the motion, and this causes the velocity of the train to diminish.

In all the above formulas involving inertia and kinetic energy, the letters W and g enter. W has been defined to be the force with which gravity acts upon the body as determined by a spring balance, and g has been defined as the acceleration of the body in a vertical fall. Both W and g are hence measures of the same force in different ways, and accordingly their ratio W/g must be constant at all places. This ratio depends only upon the quantity of matter in the body, and it is generally called 'mass' in higher works on theoretical mechanics and is designated by M. Accordingly the force of inertia may be written $F = Mf$, and the kinetic energy of a moving body may be expressed by $K = \frac{1}{2}Mv^2$. These works also use a unit for W different from that employed in this book, or they determine M by balancing it on a lever against a body of known mass. The system of units used in this book is called the 'gravitation system,' and is the one generally employed by civil, mechanical, and mining engineers. The other system of units is of special value in celestial mechanics where bodies have no weight in the sense here employed.

PROBLEMS

351. The diameter of the bore of a gun is 10 inches and the explosion of the powder exerts a pressure of 30 000 pounds per square inch on the end of the projectile, which weighs 372 pounds. If the pressure of the powder is constant and the projectile moves to the muzzle in one-hundredth of a second, compute the velocity of the projectile.

352. A body weighing 12 pounds is moving with a velocity of 100 feet per second when a constant force of 40 pounds acts upon it in the direction of motion for 7 seconds. Find the increase in velocity due to this force. Find the velocity at the end of 7 seconds.

353. For the same data find the final velocity and its direction, if the force is normal to the direction of the initial velocity.

354. A body weighing 12 pounds is moving with a velocity of 100 feet per second when a constant applied force of 40 pounds acts upon it in the direction of the motion through a distance of 22 feet. Compute the final velocity.

355. What change in the answer of the last problem will be needed if the constant force acts at an angle of 30 degrees with the initial velocity?

356. Find the direction of the final velocity v for the case of the two preceding problems.

357. An iron ball weighing 80 pounds falls vertically from a height of 4 feet and drives a nail into a plank, and the time of descent of the nail is one-sixth of a second. Compute the pressure on the nail, assuming it to be constant.

358. An iron ball weighing 80 pounds falls vertically from a height of 4 feet and drives a nail 1.5 inches into a plank. If the pressure on the nail be constant, compute its value.

359. In Fig. 128 let W_1 be 2 pounds, W_2 be 8 pounds, and the distance moved over by W_1 and W_2 in 4.5 seconds be 10.15 feet. Compute the coefficient of friction on the plane.

360. A bullet weighing $\frac{1}{4}$ pound is fired into a wooden ball weighing 28 pounds which is suspended by a string, causing it to swing and to rise in its swing through a vertical height of 4.375 inches. Compute the velocity of the bullet.

ART. 37. CENTRIFUGAL FORCE

When a body revolves around a fixed axis with uniform velocity, as in Fig. 131, the direction of its velocity is always tangential to the circle in which it revolves, and there is a tension in the arm or cord which connects it with the axis. This tension acts both on the body and on the axis, and this is called 'centripetal force' when considering its action on the body and 'centrifugal force' when speaking of its action on the axis. Its magnitude depends upon the weight and velocity of the body and upon the radius of the circle of revolution. If this force is destroyed by the breaking of the cord, the body continues its uniform motion in a straight line which is tangent to the circle at the point where the body was when the rupture occurred. If the student is not acquainted with the above facts by experience, let him try experiments with a stone tied to a string.

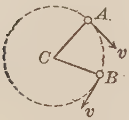

FIG. 131

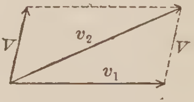

FIG. 132

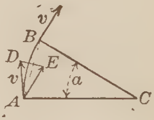

FIG. 133

The tension or centrifugal force F in the cord may be found from the inertia formula $F = W \, . \, f/g$, where W is the weight of the body, g the acceleration of it by gravity in a vertical fall, and f the acceleration due to F, for in compelling the direction of the velocity to change, the force F must overcome the inertia of the body (Art. 35). In order to find the value of f, let A and B in Fig. 133 represent two positions of the body, and a the angle included between the corresponding positions of the cord. The velocity at both A and B is v, and its direction is tangential to the circle. At A let AD and AE be drawn to represent these velocities, AE being parallel to the direction of the velocity at B; then the angle DAE is equal to a. The velocity of B relative to A is now represented by the line DE, and this will be called V; this follows from Art. 34, and may be further illustrated by

Fig. 132, where v_1 represents the velocity of a body at a certain instant, v_2 its velocity at a later instant, and V the velocity relative to the body moving in the direction v_1. Now the similar triangles ADE and CAB give $DE/AD = AB/AC$, that is $V = AB \cdot v/r$. If the angle a is very small, the chord AB will be very nearly the length of the arc, so that $AB = vt$, where t is the time in which the body passes over that distance. Accordingly, $V = v^2t/r$ and the acceleration $f = V/t = v^2/r$. Therefore

$$F = W . f/g \qquad f = v^2/r \qquad F = W . v^2/gr$$

and the last equation gives the value of the centrifugal force of the revolving body.

The last formula shows that the centrifugal force of a body revolving about an axis varies directly as its weight and the square of its velocity, and inversely as the radius of the circle of revolution. Thus if the weight is doubled, the centrifugal force is doubled; if the velocity is doubled, the centrifugal force is quadrupled; if the radius is doubled, the centrifugal force is halved. These conclusions have been verified by many experiments.

By help of the formula $F = W . v^2/gr$, numerous problems relating to centifugal force may be solved. For example, let a body weighing 2.4 pounds be attached to a string that will break under a tension of 80 pounds, and let it be required to find the velocity which the body must have in a circle of 3.11 feet radius in order that the string shall break. The value of v^2 is $gr . F/W$, or $v = (32.16 \times 3.11 \times 80/2.4)^{\frac{1}{2}} = 57.74$ feet per second. Since the circumference of the circle is $2 \times 3.1416 \times 3.11 = 19.54$ feet, it follows that the number of revolutions to be made per second is $57.74/19.54=2.95$, so that the number per minute required to break the string is about 177.

In the above discussion the body is supposed to be a particle at the distance r from the axis, but it will now be shown that the radius to be taken for any actual body is that of its center of gravity. Let n be the velocity of a point at the distance unity from the axis, then the velocity v at the distance r is $v=nr$. Let w be the weight of a particle at the distance r from the axis, then the centrifugal force of this particle is $wv^2/gr=(n^2/g)wr$. Let D in Fig. 134 represent one particle of the body, C the axis, CX and CY two direction lines at right angles to each other through C, a the angle which CD makes with CX, and x and y the distances from D to CY and CX. Then the components of the force $(n^2/g)wr$ parallel to CX and CY are $(n^2/g)wr \cos a$ and $(n^2/g)wr \sin a$; but $\cos a=x/r$ and $\sin a=y/r$, whence these components are $(n^2/g)wx$ and $(n^2/g)wy$. The sums of the components of all the forces acting on all the particles hence are $(n^2/g)\Sigma wx$ and $n^2/g\Sigma wy$. Now from the definition of center of gravity in Art. 13 and from the method of finding it in Art. 15, it follows that $\Sigma wx=Wx_1$ and $\Sigma wy=Wy_1$, where W is the total weight of the body and x_1 and y_1 are the distances of its center of gravity from OY and OX. Hence $(n^2/g)Wx_1$ and $(n^2/g)Wy_1$ are the rectangular

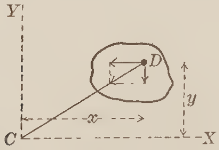

FIG. 134

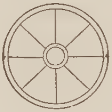

FIG. 135

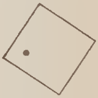

FIG. 136

components of the total centrifugal force F, and from Art. 6 its value is

$$F = (n^2/g)W(x_1{}^2 + y_1{}^2)^{\frac{1}{2}} \quad \text{or} \quad F = (n^2/g)Wr = W.v^2/gr$$

in which r is the radius of the center of gravity of the body and v is its velocity in the circle of revolution. When the axis passes through the center of gravity of a body, r and v are zero and there is no pull of the body upon its axis.

PROBLEMS

361. What is the acceleration of a particle which moves in a circle of 12 feet diameter with a velocity of 9 feet per second?

362. A particle revolves 2000 times per minute in a circle having a radius of 3 inches. Compute its velocity.

363. A body revolves in a circle of 1.5 feet radius. What velocity must it have in order that the pull upon the axis may be double the weight of the body?

364. The term 'angular velocity' is often used for the velocity of a point at a distance unity from the axis. Compute the angular velocity of a particle which revolves 1000 times per minute in a circle of 1.5 inches radius.

365. A rod 6.5 feet long and weighing 15 pounds is revolved about an axis at one end. Compute the pull on the axis when the number of revolutions per second is 47.

366. If the connection of this rod to its axis should break, in what direction will the center of gravity of the rod move, and what kind of motion will the rod as a whole have?

367. Fig. 135 shows a wheel with eight light spokes. The rim of the wheel weighs 84 pounds and its mean radius is 1.375 feet. Find the tension in each spoke when the wheel revolves 150 times per minute on a vertical axis.

368. Fig. 136 shows a square plate having an axis half-way

between one corner and the center. The plate weighs 13 pounds per square foot and the side of the square is $14\frac{1}{8}$ inches. Compute the pull on the axis when the number of revolutions per minute is 420.

369. A ball weighing 16 pounds is revolved in a vertical plane in a circle of 3 feet radius, the number of revolutions per minute being 105. Find the tension in the cord for each of its vertical positions.

370. A horizontal plane is made to revolve about a vertical axis with such speed that a point one foot from the axis has a velocity of 3.5 feet per second. How far from the axis must a body be placed on this plane, so that it may be just on the point of sliding outward, the coefficient of friction between the body and the plane being 0.65?

ART. 38. REVOLVING BODIES

A wheel like Fig. 135 can only be put in motion around its axis by applied forces which overcome its inertia, and when in motion it can only be stopped by opposing forces. In other words work must be performed upon a wheel in order to cause it to revolve; when in motion it has stored within it an amount of kinetic energy equivalent to that work; when stopped, this energy performs work against the opposing resistances. A revolving fly-wheel may hence be regarded as a machine in which energy is stored.

Let w be the weight of any particle in a revolving body, z its distance from the center of the axis, and v its velocity. Its kinetic energy then is $w \cdot v^2/2g$, and the total kinetic energy in the wheel is $\Sigma wv^2/2g$, in which the summation is to include all the particles of the body. Now let W be the total weight of the body, R the radius of a circle upon the circumference of which this weight would have to be concentrated in order that its kinetic energy should be equal to $\Sigma wv^2/2g$, and V_R the velocity of the end of this radius. The kinetic energy of the weight W moving with the velocity V_R is $W.V_R{}^2/2g$. The value of R is found by equating these energies and noting that the velocities vary directly as their distances from the axis; thus

$$W.V_R^2/2g = \Sigma wv^2/2g \qquad V_R = v\,.\,R/z \qquad R^2 = (1/W)\Sigma wz^2$$

The value of R determined from the last equation is called the 'radius of gyration' of the revolving body; it is the radius of a circumference where the entire weight of the body might be put and have the same kinetic energy as the actual distributed mass when the body revolves around an axis. This circumference may be called the 'gyration circle.'

The values of the radii of gyration for regular solids are not easily determined without the help of a branch of mathematics called calculus, and hence the following are stated without demonstration, the axis in all cases passing through the center of the solid and coinciding with its geometric axis:

Cylinder with radius r,	$R^2 = \frac{1}{2}r^2$
Hollow cylinder with radii r_1 and r_2,	$R^2 = \frac{1}{2}(r_1{}^2 + r_2{}^2)$
Sphere with radius r,	$R^2 = \frac{2}{5}r^2$
Rectangular prism with sides b and d,	$R^2 = \frac{1}{12}(b^2 + d^2)$

By the help of these values of R^2 the kinetic energy K, stored in one of these bodies when revolving about its axis, may be found by

$$K = W.\,V_R^2/2g \qquad \text{and} \qquad V_R = 2\pi R\,.\,N$$

where N is the number of revolutions per second, π is 3.1416, V_R is the velocity of the gyration circle whose radius is R, and W is the weight of the body measured by a spring balance at the place where the acceleration of gravity is g.

For example, take a wheel or millstone of 3.2 feet diameter and of uniform thickness which weighs 800 pounds, and let it be required to find the kinetic energy stored in it when making 750 revolutions per minute. Here the radius of gyration $R=0.707\times 1.6=1.131$ feet, and $V_R=2\times 3.1416\times 1.131\times 750/60=88.83$ feet per second. Then $K=800\times 88.83^2/64.32=98\,140$ foot-pounds, which is the kinetic energy stored in the wheel. This energy might be utilized in lifting a weight attached to a rope which winds around the shaft of the wheel, and the height to which a weight of 400 pounds could be raised would be $98\,140/400=245.3$ feet. If this work is done in 30 seconds, the wheel will deliver during this time $98\,140/30\times 550=5.95$ horse-powers.

A constant force P applied to a crank on a wheel at a distance r_1 from the axis of revolution will cause a uniformly accelerated motion of the wheel and store in it the above kinetic energy K, if no work be expended in overcoming resistances other than that of inertia. Let N_1 be the number of revolutions performed under this constant force; then the work imparted to the wheel is $P\times 2\pi r_1 N_1$, and this equals the stored kinetic energy K. When the wheel is stopped by a constant force P acting at the distance r_1 from the axis, the kinetic energy K does work against this resistance, and also $P\times 2\pi r_1 N_1=K$. For instance, let the radius of the shaft in the last paragraph be 0.12 feet, then the number of revolutions performed while the wheel is coming to rest is $N_1 = 98\,140/(400 \times 6.283 \times 0.12) = 338$ nearly.

PROBLEMS

371. The circumference of a wheel of 5 feet diameter has a velocity of 350 feet per second. What is the velocity of a point distant 2.4 feet from the axis?

372. A hollow cylinder has an outer diameter of 8 feet and an inner diameter of 7 feet. Find the radius of gyration with respect to its axis.

373. A plate 6 inches square revolves about an axis normal to it and passing through its center. Compute the radius of gyration, and find the velocity of the end of that radius when the number of revolutions per second is 75.

374. If this plate weighs 120 pounds, find the kinetic energy stored in it and the horse-power which it can develop when stopped in $1\frac{1}{4}$ seconds.

375. Whirl a bucket of water rapidly around its vertical axis and explain the phenomena which are observed.

376. Why are fly-wheels placed upon the revolving shafts of steam-engines, and why are none used on the shafts of electric motors?

377. When a body is caused to revolve by a constant force P at the end of a crank, show that $Pr_1^2 = WR^2$.

378. A grindstone weighing 150 pounds per cubic foot has a thickness of 6 inches and a diameter of 3 feet. How much energy is stored in it when it makes 300 revolutions per minute?

379. If a constant force of 18 pounds be applied to the crank of this grindstone at 8 inches from the axis, how many revolutions must be performed in order to store that kinetic energy in it?

380. A ball weighing 2 pounds is attached to a string and whirled around 6.5 times per second in a vertical circle of 4.7 feet radius. If the string breaks when it is horizontal, in

what direction does the ball move and what is its kinetic energy in foot-pounds?

ART. 39. ROLLING BODIES

When a wheel or cylinder is rolling on a plane with an absolute velocity v parallel to the plane, as in Fig. 137, each point on its circumference must have the velocity v relative to the axis, for if r be the radius of that circumference, one revolution occurs when the axis has moved the distance $2\pi r$, and each velocity is equal to this distance divided by the same time. The relative velocity of the circumference of gyration whose radius is R is less than v and is given by $V_R = v \,.\, R/r$. Upon this circumference the entire weight of the body may be concentrated and have the same kinetic energy with respect to the axis as the actual distributed weight (Art. 38).

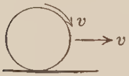

FIG. 137

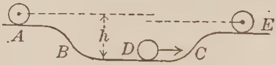

FIG. 138

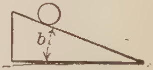

FIG. 139

The total kinetic energy stored in a body which has motions of both translation and rotation may be ascertained by finding the sum of the kinetic energies due to these motions, and it now will be proved that

$$K = W.(v^2 + V_R^2)/2g$$

is its value. To do this, consider that in one revolution of the wheel the absolute velocity of a point which has the relative velocity V_R constantly varies, it being $v + V_R$ when the point is vertically above the axis and $v - V_R$

when it is vertically below the axis. The mean of the kinetic energies in these two positions is proportional to one-half of $(v+V_R)^2+(v-V_R)^2$, that is to $v^2+V_R^2$. The same result follows for any two points upon this circumference at opposite ends of a diameter, for the relative velocity at one end is inclined forward and that at the other end has the same inclination backward. Hence the total kinetic energy of a rolling body is the sum of the kinetic energies of translation and rotation.

When a body rolls on a horizontal plane by virtue of its fall from a height h, as in Fig. 138, the potential energy Wh is equal to the above kinetic energy K if no work has been expended in overcoming friction or air resistance. Accordingly, noting the relation between V_R and v, there results

$$h=(v^2+V_R^2)/2g \qquad \text{or} \qquad v^2=2gh/(1+R^2/r^2)$$

which shows that the velocity of translation of a rolling body due to a vertical fall under gravity is less than that where translation only occurs, which was the case discussed in the last chapter. For example, consider a solid rolling wheel or cylinder for which R^2 is $\frac{1}{2}r^2$; then $v=(\frac{4}{3}gh)^{\frac{1}{2}}=0.815(2gh)^{\frac{1}{2}}$, so that the velocity of translation is $81\frac{1}{2}$ percent of that which would occur if there were no rotation around the moving axis. For the sphere R^2 is $\frac{2}{5}r^2$, and hence $v=(\frac{10}{7}gh)^{\frac{1}{2}}=0.845(2gh)^{\frac{1}{2}}$, so that the velocity of the center of the sphere is $84\frac{1}{2}$ percent of that which would occur if no rotation existed. The height h is here, as always, the vertical distance through which the center of gravity of the body falls.

The time required for a body to roll down an inclined plane is greater than that determined for sliding in Art.

31. Let l be the length of the plane in Fig. 139, b its angle of inclination, h its height, and v the velocity of translation at the foot of the plane. The body starting from rest and having uniformly accelerated motion, the distance l equals $\frac{1}{2}vt$ (Art. 29), or $t=2l/v$. Now the above equation gives v in terms of h, and replacing h by $l \sin b$, there is found

$$t=\left(2l(1+R^2/r^2)/g \sin b\right)^{\frac{1}{2}}$$

as the time of fall for a rolling body, whereas for a sliding body the time is $(2l/g \sin b)^{\frac{1}{2}}$.

Owing to friction and air resistance, the actual velocity of a body rolling down a plane is less and the time of its descent is greater than the above formulas give. Also when a sphere rolls down a curve AB and then advances upon a horizontal plane BC, as in Fig. 138, its velocity is less than above given, so that it cannot rise on the curve CE to the same level from which it started. If h' be the difference in elevation of the two positions A and E, the quantity Wh' is the work expended in overcoming the friction and air resistance, E being the highest portion to which the ball can rise.

Rolling friction is always less than sliding friction because the rolling motion lifts the projections of the surface of the ball out of the depressions in the surface of the plane, whereas in sliding these projections must be broken off. Sliding friction is about five or six times as great as rolling friction, as may be seen by comparing the coefficients of sliding friction in Art. 19 with those of traction or rolling friction in Art. 22. For this reason vehicles with wheels are preferred for travel on all surfaces except snow.

PROBLEMS

381. Let the velocity for Fig. 137 be 9.5 feet per second and the radius of the wheel be 3.2 feet. Compute the absolute velocity of a point 1.6 feet above the axis, and also of a point 1.6 feet below the axis.

382. For the same data compute the absolute velocities of these points after the line joining them has become horizontal.

383. A sphere rolls on a horizontal plane with uniform velocity and travels through a distance of 37.6 feet in 1.5 seconds. To what height can this sphere lift itself on an inclined plane?

384. Show that the time required for a sphere to roll down an inclined plane is about 18 percent greater than the time for a sliding body.

385. Compute the time required for a circular hoop to roll down an inclined plane, the length of the plane being 65 feet and its height 25 feet, if there is no friction.

386. Solve the same problem, taking the coefficient of rolling friction as 0.06. (See Art. 31.)

387. A circular hoop rolling on a plane has a kinetic energy of 80 foot-pounds. What part of this is due to translation and what part to rotation?

388. A hollow cylinder with radii of 4 and 1.5 feet rolls on a plane and has a kinetic energy of 800 foot-pounds. What part of this is due to translation and what part to rotation?

389. A hollow sphere of gold has the same weight and the same outer diameter as a hollow sphere of silver, and the surfaces are painted so that it is impossible to see the metals. How can it be ascertained which sphere is the gold one, by rolling both spheres down an inclined plane?

390. A rolling sphere weighing 18 pounds has a velocity of 30 feet per second at the foot of an inclined plane, and it rises on that plane to a vertical height of 16 feet. Compute the work lost in friction.

ART. 40. PENDULUMS

A simple pendulum consists of a small body attached to a light rod or cord which is fastened to a support, so that oscillations may occur in a vertical plane when the bob is drawn to one side of the vertical line and then released. These oscillations are observed to occur in nearly the same time whether the angle of swing be large or small. Let a be the angle included between the vertical line and the extreme position AB in Fig. 140, and l the radius AB or AC. Then it is shown in advanced works on mechanics that the time of oscillation from the position AB to the position AD is

$$t=(1+\tfrac{1}{4}\sin^2\tfrac{1}{2}a+\tfrac{9}{64}\sin^4\tfrac{1}{2}a+\tfrac{25}{256}\sin^6\tfrac{1}{2}a+\ldots)\pi\sqrt{l/g}$$

provided the rod or cord has no weight and the bob be a mere particle at the end of the radius r. For $a=5°$ this formula becomes $t=1.00048\pi(l/g)^{\frac{1}{2}}$, and hence for this and smaller angles the time of oscillation is given with sufficient precision by regarding the quantity in the parenthesis as unity, or

$$t=\pi\sqrt{l/g} \qquad \text{and} \qquad l=gt^2/\pi^2$$

are the usual formulas for the simple pendulum. These show that the times of oscillation are proportional to the square roots of the lengths. The mean length of a pendulum which oscillates in one second in the United States of America is $l=g/\pi^2=32.16/(3.1416)^2=3.259$ feet or about 39.1 inches; hence the length of a pendulum which oscillates in half a second is 9.78 inches, and the length of one which oscillates in two seconds is about 13.0 feet.

To apply the above formula to the case of the actual pendulum, the size of the bob and rod must be taken into account as follows. Let r be the distance from the axis of suspension to the center of gravity of the bob and rod, and let R be the radius of gyration of bob and rod with respect to that center of gravity. Then the above formula for the mathematical simple pendulum applies to the actual pendulum if l be replaced by $r+R^2/r$. For example, take the simple case of a rectangular plate of uniform thickness and material, as shown in Fig. 141, which oscillates about an axis near its upper end. Let the width of the plate be 3 inches, its length 18 inches, and the distance from the axis of suspension to the center of gravity be 8 inches. From the last article the value of R^2 is $\frac{1}{12}(3^2+18^2) = 27.75$, and hence $l = 8+3.469 = 11.3469$ inches $=0.9456$ feet; then from the above formula the time of one oscillation of the plate is $t=0.54$ seconds.

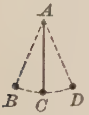

FIG. 140

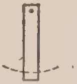

FIG. 141

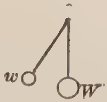

FIG. 142

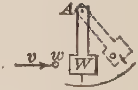

FIG. 143

By determining the times of oscillation of the same pendulum in different parts of the earth, the relative values of the acceleration g in different latitudes have been ascertained. Thus, let t_1 be the time of oscillation of a pendulum at the equator, t_2 its time at New York, and g_1 and g_2 the corresponding values of the acceleration; then from the above formula $g_1 t_1^2$ equals $g_2 t_2^2$, whence $g_2 = g_1(t_1/t_2)^2$. The value g_1 for the equator is found by accurately determining the length l_1 by the method of the last paragraph for a pendulum which oscillates there in one second and then $g_1 = \pi^2 l_1$. In this manner the values of g stated in Art. 30 were ascertained.

Pendulums were used by Newton for determining the facts and laws of impact in the manner indicated in Fig. 142, where a weight W is struck by another weight w. By virtue of its fall the body w acquires a horizontal velocity v, and if it moves away with W, the weight $w+W$ has the velocity v_1. Newton showed by his experiments that in this case wv always equals $(w+W)v_1$. This law is the foundation of the theory of the subject of impact, which can here be illustrated only for the following case.

On this principle the ballistic pendulum shown in Fig. 143 is constructed, this being an apparatus for determining the velocity of a bullet just before it strikes the wooden block, of the pendulum. The bullet penetrates the wooden block, and the impact causes the pendulum to swing out of the vertical, while the angle of swing is observed on a graduated arc. Let w and W be the weights of the ball and pendulum, r_1 the distance from the axis to the line of direction of the horizontally moving bullet, r the distance from the axis to the center of gravity of the pendulum and imbedded bullet, t the time of one oscillation of the pendulum after the bullet is in it, and a the extreme angle of deviation from the vertical. Then works on higher mechanics prove that

$$v = \frac{w+W}{w} \cdot \frac{r}{r_1} \cdot \frac{2gt}{\pi} \cdot \sin\tfrac{1}{2}a$$

gives the velocity of the bullet. This expression is a very accurate one, since it is true whether energy be expended in heat or not, and since t can be very precisely determined by counting the number of oscillations which occur during several minutes.

This article and the preceding one give some things which the student is asked to accept without demonstration on account of the limited space available for this volume. The field of mechanics is a vast one, and this book presents merely the fundamental elements. The student who solves the four hundred problems here given will, however, have constructed a broad and strong foundation for his future studies in theoretical and applied mechanics.

PROBLEMS

391. A clock pendulum swings from one side of the vertical to the other $160\frac{1}{2}$ times in exactly two minutes. Find the time of one oscillation, and the length of the corresponding simple pendulum.

392. Compute the time of oscillation of the simple pendulum when the angle of swing on one side of the vertical is 15 degrees.

393. What is the length of a simple pendulum which makes 113 oscillations in 355 seconds?

394. While a man is running half a mile, a stone attached to a string 72 inches long makes 129 oscillations in a vertical plane. What is his record?

395. A circular plate of uniform thickness and 18 inches diameter has a small hole through it at a distance of 6 inches from the center. Find the time of oscillation when it swings on a horizontal axis through the hole.

396. The value of g for sea-level at the poles of the earth is 9.8322 meters per second per second, and its value for an elevation of 4000 meters at the equator is 9.7683 meters per

second per second. The length of the seconds pendulum at the first place being 0.9962 meters, what is its length at the second place?

397. A delicate spring balance shows the weight of a body at the equator, on a mountain 4000 meters high, to be 37.45 grams. What will be its weight at sea-level at the north pole?

398. The radius of the earth's equator is 6378 kilometers. What must be the time of the earth's revolution on its axis in order that the centrifugal force at the equator may overcome the force of gravity?

399. A ballistic pendulum weighing 3000 pounds is used to determine the velocity of a ball weighing 0.35 pounds, the ball passing into a clay pocket whose center is at the same distance from the axis as the center of gravity of the pendulum. The angle of swing is seen to be 15° 30′, and it is then noted that the number of oscillations made in one minute is 39½. Compute the velocity of the ball.

400. A rifle weighing 21 pounds is suspended in a horizontal position by a cord at its center of gravity. A bullet weighing 2.1 ounces is then fired, and this causes the rifle to move backward. If the velocity of the bullet is 250 feet per second, what is the initial backward velocity of the rifle? What constant force, acting for one second after the explosion, will bring the rifle to rest?

APPENDIX

Answers to Problems

The following are answers to some of the problems given in the preceding pages, the number of the problem being in parenthesis. The student should not consult the answer to a problem until he has finished its solution, for a knowledge of the answer renders him liable to work only for the purpose of obtaining that numerical result. The first thing to be done in solving a problem is to thoroughly understand it, and for that purpose a diagram representing the data should always be drawn. Then a mental estimate of the answer, based on experience, should be made and be recorded. The solution then proceeds, after the manner of those given in the text, and on its completion the numerical work should be checked or the results be verified by an independent process. Lead pencils and scraps of paper should never be used, but the solutions should be made in ink in a special book devoted to that purpose.

(4) 87.5 pounds; 69.5 pounds. (5) 10.4 pounds. (7) 43 pounds. (8) 9.97 pounds. (11) 180 degrees. (14) 85.7 pounds; 51.5 pounds. (17) 280 pounds. (19) 23.33 pounds. (21) +25 pounds. (23) −100 kilograms for vertical. (28) 100 pounds; 110° 18′. (30) 65° 46′. (32) 14.1 pounds. (33) +63.0 pounds. (38) 1000 pounds. (44) $\frac{1}{2}F \sec\frac{1}{2}a$. (45) $V = 24.7$ and $P = 48.7$ kilograms. (49) $S_1 = 50\,900$ and $S_2 = 56\,600$ pounds. (51) $b = 53°$. (60) 60° 12′ and 47° 30′;

77.4 and 91.1 pounds. (62) 30 grams. (67) 1 to 3. (69) +54 kilograms. (76) 10 and 50 pounds. (77) 127 and 73 pounds. (82) −60 pounds at 7.33 feet from left end. (86) $\frac{3}{4}P$. (92) Clockwise. (97) Yes. (102) 115.5 pounds. (103) 72.7 pounds. (104) 28.92 inches. (118) $R = 8400$ pounds. (122) 9.33 feet from second sphere. (127) $\frac{1}{2}\frac{3}{8}$ths of the length from the left end. (130) 2892 miles. (131) 2.73 feet from the left end. (137) 1.36 feet from the load. (138) 5 pounds. (145) 2.20 inches to right of AB and 2.20 inches above AF. (146) 0.34 inches from center. (150) 1 to 10. (152) 10 pounds. (153) 11.55 pounds. (161) $11\frac{1}{4}$ inches. (163) 12.96 feet or more. (164) 109.2 pounds. (169) 5.70 feet from A. (173) $R = 25.6$ pounds at 1.59 inches from center of base. (175) 21° 10′. (178) $F = 1.44$ kilograms. (182) $n = 0.383$. (184) 11° 16′. (189) 16.18 pounds. (190) 28.62 pounds. (192) 9.11 pounds. (194) 2400 pounds. (195) 79° 26′. (202) 65.77 horse-powers. (203) 3000 foot-pounds. (207) 4 feet. (208) 111 metric horse-powers. (211) 5.87 horse-powers. (214) 3125 pounds. (219) 54 125 foot-pounds. (220) 816 horse-powers. (221) 25, $66\frac{2}{3}$, 150, 400 pounds. (224) 73.33 pounds. (230) 1.8 horse-powers. (234) 16 inches. (235) 10.16 inches. (239) 9.83 pounds. (240) 27.92 pounds. (243) 58.3 pounds. (248) 25.0 horse-powers. (249) 0° 17′. (252) 19.9 pounds. (253) 32.4 pounds. (255) 7.64 pounds. (261) 14 pounds. (263) $P = \frac{1}{16}W$. (265) 10 pounds. (277) 24.2 percent. (278) 704.8 foot-pounds. (280) 1.12 horse-powers. (283) 33.3 feet per second per second. (287) 1.56 feet. (289) 5.69 seconds. (291) 4.66 seconds. (293) 0.072 and 0.144 seconds. (297) 748 feet. (301) 2.28 seconds. (305) 40.7 feet per second. (306) 8° 45′. (313) 339 horse-powers. (317) 941.4 feet. (319) 9.41 pounds. (320) 7160 foot-pounds. (323) 235.4 feet. (326) 1680 feet per second. (330) About 46 miles per hour. (333) 26.63 miles per hour; 10° 43′ to right of northeast. (336) 4.69 seconds. (337) 4.24 miles per hour from northwest. (340) 9.20 miles per hour. (342) 1286 pounds. (345) 197.2 pounds. (350)

3.86 feet per second; 0.231 feet. (351) 2037 feet per second. (352) 750.4 and 850.4 feet per second. (353) 7.57 feet per second at angle of 82° 25′ with initial velocity. (358) 2560 pounds. (363) 9.81 feet per second. (365) 132,200 pounds. (369) 164.5 pounds. (370) 1.71 feet. (372) 3.76 feet. (374) 17 300 foot-pounds; 25.1 horse-powers. (378) 9160 foot-pounds. (383) 6.98 feet. (385) 4.58 seconds. (386) 4.96 seconds. (388) 509.5 and 290.5 foot-pounds. (393) 32.16 feet. (394) 2 minutes 55 seconds. (395) 0.57 seconds. (398) Seventeen times as fast as at present.

TRIGONOMETRIC FUNCTIONS

Read down for Sines and Tangents

	SINES					TANGENTS					
D	0′	15′	30′	45′	60′	0′	15′	30′	45′	60′	D
0	000	04	09	13	17	000	04	09	13	17	89
1	17	22	26	31	35	17	22	26	31	35	88
2	35	39	44	48	52	35	39	44	48	52	87
3	52	57	61	65	70	52	57	61	66	70	86
4	70	74	78	83	87	70	74	79	83	87	85
5	87	92	96	*0	05	87	92	96	*1	05	84
6	105	09	13	18	22	105	10	14	18	23	83
7	22	26	31	35	39	23	27	32	36	41	82
8	39	43	48	52	56	41	45	49	54	58	81
9	56	61	65	69	74	58	63	67	72	76	80
10	74	78	82	87	91	76	81	85	90	94	79
11	91	95	99	*4	08	94	99	*3	08	13	78
12	208	12	16	21	25	213	17	22	26	31	77
13	25	29	33	38	42	31	35	40	45	49	76
14	42	46	50	55	59	49	54	59	63	68	75
15	59	63	67	71	76	68	73	77	82	87	74
16	76	80	84	88	92	87	91	96	*1	06	73
17	92	97	*1	05	09	306	11	15	20	25	72
18	309	13	17	21	26	25	30	35	39	44	71
19	26	30	34	38	42	44	49	54	59	64	70
20	42	46	50	54	58	64	69	74	79	84	69
21	58	62	67	71	75	84	89	94	99	*4	68
22	75	79	83	87	91	404	09	14	19	24	67
23	91	95	99	*3	07	24	30	35	40	45	66
24	407	11	15	19	23	45	50	56	61	66	65
25	23	27	31	34	38	66	72	77	82	88	64
26	38	42	46	50	54	88	93	99	*4	10	63
27	54	58	62	66	69	510	15	21	26	32	62
28	69	73	77	81	85	32	37	43	49	54	61
29	85	89	92	96	*0	54	60	66	72	77	60
30	500	04	08	11	15	77	83	89	95	*1	59
31	15	19	22	26	30	601	07	13	19	25	58
32	30	34	37	41	45	25	31	37	43	49	57
33	45	48	52	56	59	49	56	62	68	75	56
34	59	63	66	70	74	75	81	87	94	*0	55
35	74	77	81	84	88	700	07	13	20	27	54
36	88	91	95	98	*2	27	33	40	47	54	53
37	602	05	09	12	16	54	60	67	74	81	52
38	16	19	23	26	29	81	88	95	*3	10	51
39	29	33	36	39	43	810	17	24	32	39	50
40	43	46	49	53	56	39	47	54	62	69	49
41	56	59	63	66	69	69	77	85	93	*0	48
42	69	72	76	79	82	900	08	16	24	33	47
43	82	85	88	92	95	33	41	49	57	66	46
44	95	98	*1	04	07	66	74	83	91	*0	45
D	60′	45′	30′	15′	0′	60′	45′	30′	15′	0′	D
	COSINES					COTANGENTS					

	SINES					TANGENTS					
D	0′	15′	30′	45′	60′	0′	15′	30′	45′	60′	D
45	707	10	13	16	19	1.000	09	18	27	36	44
46	19	22	25	28	31	36	45	54	63	72	43
47	31	34	37	40	43	72	82	91	*1	11	42
48	43	46	49	52	55	1.111	20	30	40	50	41
49	55	58	60	63	66	50	61	71	81	92	40
50	66	69	72	74	77	92	*2	13	24	35	39
51	77	80	83	85	88	1.235	46	57	68	80	38
52	88	91	93	96	99	80	92	*3	15	27	37
53	99	*1	04	06	09	1.327	39	51	64	76	36
54	809	12	14	17	19	76	89	*2	15	28	35
55	19	22	24	27	29	1.428	41	55	69	83	34
56	20	31	34	36	39	1.48	50	51	53	54	33
57	39	41	43	46	48	54	55	57	58	60	32
58	48	50	53	55	57	60	62	63	65	66	31
59	57	59	62	64	66	66	68	70	71	73	30
60	66	68	70	72	75	73	75	77	79	80	29
61	75	77	79	81	83	80	82	84	86	88	28
62	83	85	87	89	91	88	90	92	94	96	27
63	91	93	95	97	99	96	98	*1	03	05	26
64	99	*1	03	04	06	2.05	07	10	12	14	25
65	906	08	10	12	14	14	17	19	22	25	24
66	14	15	17	19	21	25	27	30	33	36	23
67	21	22	24	26	27	36	38	41	44	48	22
68	27	29	30	32	34	48	51	54	57	61	21
69	34	35	37	38	40	61	64	67	71	75	20
70	40	41	43	44	46	75	79	82	86	90	19
71	46	47	48	50	51	90	95	99	*3	08	18
72	51	52	54	55	56	3.08	12	17	22	27	17
73	56	58	59	60	61	27	32	38	43	49	16
74	61	62	64	65	66	49	55	61	67	73	15
75	66	67	68	69	70	73	80	87	94	*1	14
76	70	71	72	73	74	4.01	09	17	25	33	13
77	74	75	76	77	78	33	42	51	61	70	12
78	78	79	80	81	82	70	81	92	*3	14	11
79	82	82	83	84	85	5.14	27	40	53	67	10
80	85	86	86	87	88	67	82	98	64		9
81	88	88	89	90	90	6.31	50	69	90		8
82	90	91	91	92	93	7.12	35	60	80		7
83	93	93	94	94	95	8.14	45	78			6
84	95	95	95	96	96	9.51	93				5
85	96	97	97	97	98	11.43					4
86	98	98	98	98	99	14.30					3
87	99	99	99	99	99	19.08					2
88	99	*0	00	00	00	28.64					1
89	100	00	00	00	00	57.29					0
D	60′	45′	30′	15′	0′	60′	45′	30′	15′	0′	D
	COSINES					COTANGENTS					

Read up for Cosines and Cotangents

INDEX

Absolute velocity, 137
Accelerated motion, 115, 146
Acceleration, 115, 141, 143, 163
Answers to problems, 167
Applied forces, 72
    mechanics, 7, 165
Axioms, 11, 30, 136
Axis, 29, 33, 38, 53, 154

Balances, 99, 148
Ballistic pendulum, 164

Center line, 38
Center of gravity, 51–71
    of mass, 53
Centrifugal force, 149
Centripetal force, 149
Centroid, 53
Coefficient of friction, 77, 78
    of traction, 90
Components of forces, 8, 14
Composition of forces, 19
    of velocities, 136
Concurrent forces, 7–28, 33, 42
Conditions of equilibrium, 21, 34, 44, 49
Couples, 40, 73

Definitions, 7, 29, 51, 72, 115
Differential pulley, 110

Effect of forces, 7, 11, 13, 111
Efficiency of a machine, 111
Energy, 127
Equilibrium, 9, 21, 33, 40, 44, 49, 99, 103, 106
Experience, 11, 12, 119, 141
Experiments, 11, 30, 51, 59, 100, 150

Falling bodies, 115–140
Force, 7, 8, 72, 148
Force, centrifugal, 149
Force of friction, 73, 76
    gravity, 119
    inertia, 141, 148
    traction, 90
Forces in space, 48
Friction, 72, 73, 75, 88, 90, 160
Friction of air, 121
    on inclined plane, 102, 124
Fulcrum, 94

Gravitation units, 148
Gravity, acceleration of, 120, 163
    force of, 8, 51, 84
    work of, 84, 129

Helix, 105
Horizontal components, 15, 18
    forces, 14
Horse-power, 86, 91

Impact, 164
Inclined plane, 81, 90, 101, 105, 112, 123
Inertia, 72, 164

Kinetic energy, 128, 146, 158

Laws of friction, 75
    of mechanics, 11, 30, 136
Lever arms, 31, 94
Levers, 94, 98, 112
Lines, centers of gravity of, 54

Machines, 94–114
Mass, 148
Mechanics, defined, 7
Moment of a couple, 40
    of a force, 31
Moments, principle of, 33, 37
Motion, 8, 115
Motion, of projectiles, 131
    rotary, 29, 154
    under gravity, 115–140

Neutral equilibrium, 63
Non-concurrent forces, 43
Normal pressure, 76

Oblique fall, 123

Parallel forces, 29–50
Parallelogram of forces, 24
  of velocities, 136
Pendulums, 162
Pitch of a screw, 105
Potential energy, 126
Power, 86
Pressure, 7, 72
Projectiles, 131
Pulleys, 108, 113

Radius of gyration, 155
Reaction, 12, 72
Relative velocity, 138
Resisting forces, 72–84, 90
Resolution of forces, 26
Resultant of forces, 17, 36
Revolving bodies, 154
Roberval's balance, 100
Rolling bodies, 158
Rotation, 29, 31, 40, 67, 141–166

Scales, weighing, 98, 99
Screws, 105
Sliding friction, 75, 78, 80, 89, 160
Stable equilibrium, 63
Stability against rotation, 67
  against sliding, 80
Symmetry, axis of, 53

Tension, 7, 72
Traction, 72, 90
Translation, 29
Trigonometric functions, 170

Uniform motion, 115
Unit of force, 8
  of time, 116
  of work, 85
Units, gravitation system, 148
Unstable equilibrium, 63

Velocities, composition of, 136
Velocity, 115, 138
Vertical fall, 119
  forces, 14, 15, 18

Wagon shafts, 93
Wedge, 103
Weighing scales, 98, 99
Weight, 8, 84, 115, 119
Wind pressure, 69
Work, 84–93, 112

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